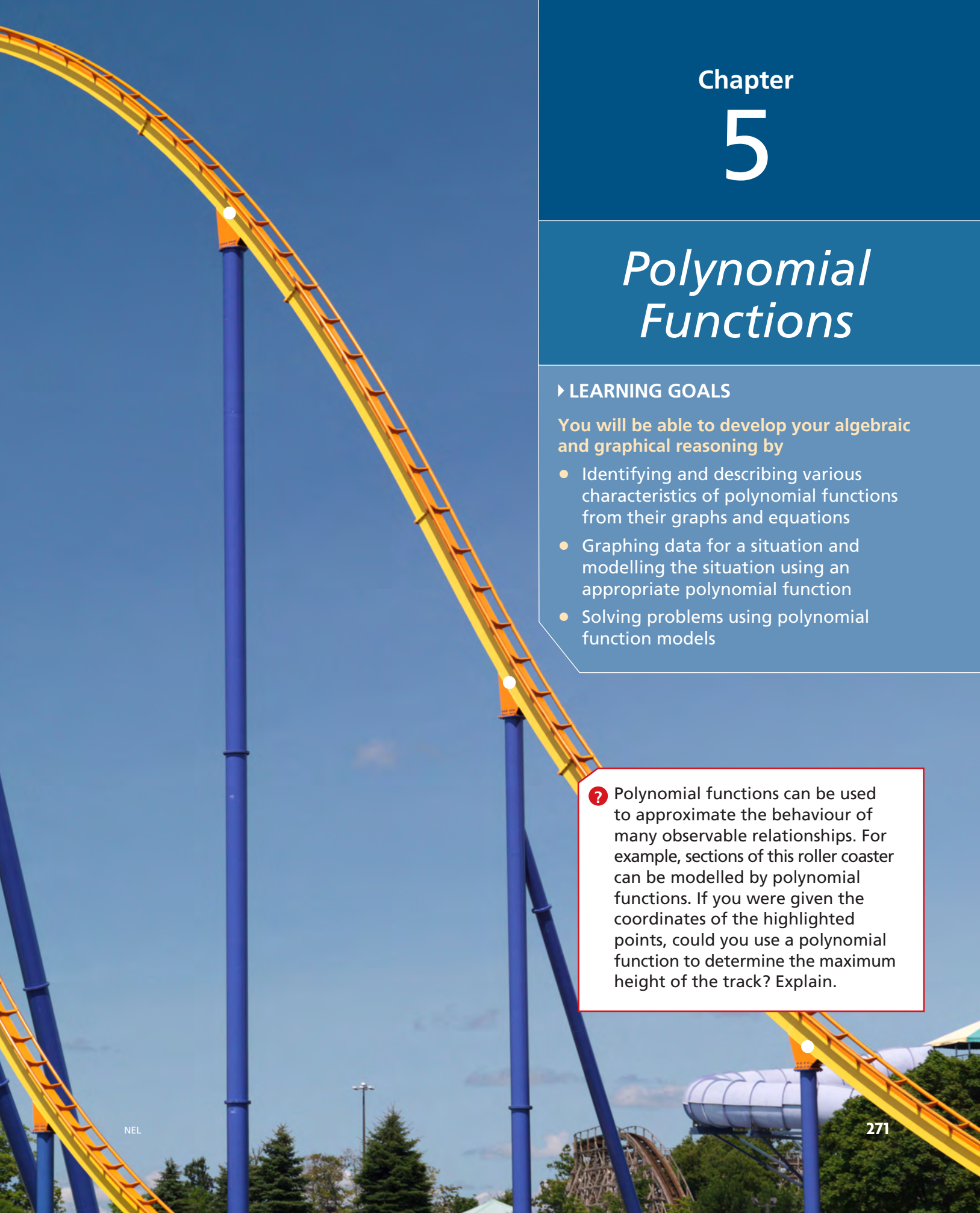




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NEL



Chapter

5

Polynomial Functions

► LEARNING GOALS

You will be able to develop your algebraic and graphical reasoning by

- Identifying and describing various characteristics of polynomial functions from their graphs and equations
- Graphing data for a situation and modelling the situation using an appropriate polynomial function
- Solving problems using polynomial function models

? Polynomial functions can be used to approximate the behaviour of many observable relationships. For example, sections of this roller coaster can be modelled by polynomial functions. If you were given the coordinates of the highlighted points, could you use a polynomial function to determine the maximum height of the track? Explain.

Staging a Profitable Production

After participating in the Festival théâtre jeunesse de l'Alberta, Patrice and his friends formed a drama club. They asked for help from l'UniThéâtre's treasurer for their budget. The treasurer determined the revenue, R , that the club can expect to generate for its next production as a function of the ticket price, t . This relationship is shown in the following table.

Ticket Price, t (\$)	Revenue, R (\$)
0	0
1	350
2	600
3	750
4	800
5	750
6	600
7	350
8	0

The cost to produce a free show is \$650. However, increasing the quality of the show will increase the cost to stage it. To cover the higher cost, the drama club can increase the ticket price, requiring the audience to pay for the performance. For each \$1 increase in the ticket price, the club can spend \$25 more on staging the production.



? What range of ticket prices will earn Patrice's drama club a profit for the production?

- A. Create a **scatter plot** to show the revenue data, with ticket price, t , on the x -axis and revenue, R , on the y -axis. Describe the trend you see. What type of function might model this trend?
- B. Determine a possible equation for the revenue function.
- C. Make a table of values to show the relationship between the cost of the production, C , and the ticket price, t .
- D. Graph the cost function you determined in part C on the same grid. Describe the trend you see. What type of function might model this trend?
- E. Determine a possible equation for the cost function.
- F. Are the revenue function and the cost function **polynomial functions**? Explain.
- G. To earn a profit, the revenue must be more than the cost of staging the production. Use your graphs or equations to determine the range of ticket prices necessary to earn a profit.

scatter plot

A set of points on a grid, used to visualize a relationship or possible trend in the data.

polynomial function

A function that contains only the operations of multiplication and addition with real-number coefficients, whole-number exponents, and two variables; for example,

$$f(x) = 5x^3 - 3x + 7$$

which can also be written as

$$f(x) = 5(x)(x)(x) + (-3)x + 7$$

WHAT DO You Think?

Decide whether you agree or disagree with each statement. Explain your decision.

1. Polynomial functions can be used to model many real-life situations.
2. Polynomial models can be used to predict accurately data points that have not been measured.
3. Polynomial functions can be graphed as either straight lines or parabolas.

5.1

Exploring the Graphs of Polynomial Functions

YOU WILL NEED

- graphing technology

GOAL

Identify characteristics of the graphs of polynomial functions.

EXPLORE the Math

Naomi researched **polynomial functions**. She learned that they have been studied for hundreds and possibly thousands of years. Originally polynomial functions were appreciated for their simplicity, because they contain only the operations of multiplication and addition.

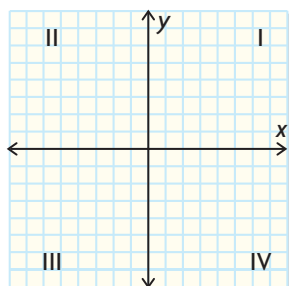
During her research, Naomi decided to investigate how the graphs of polynomial functions are related to the degree of the functions. She remembered many of the characteristics of linear functions and confirmed these characteristics using graphing technology.

end behaviour

The description of the shape of the graph, from left to right, on the coordinate plane.

Communication Tip

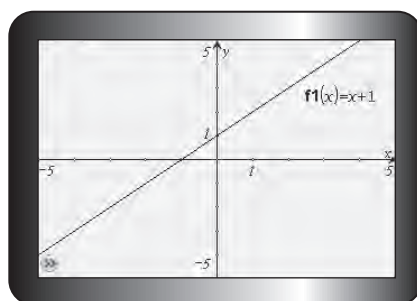
Cartesian grids are divided into four quadrants by the x -axis and y -axis. Quadrants are identified using roman numerals, from I to IV, starting from the top right and progressing counter-clockwise around the origin.



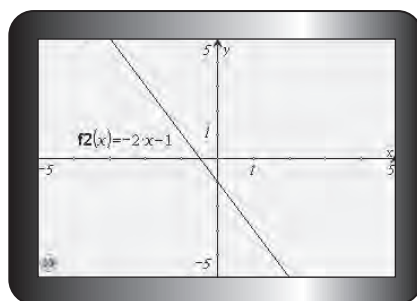
Polynomial Function

Degree 1

Graph of $f(x) = x + 1$:



Graph of $f(x) = -2x - 1$:



Common Characteristics

Functions of degree 1

Number of x -intercepts: 1

Number of y -intercepts: 1

End behaviour :

Line extends either from quadrant III to quadrant I, or from quadrant II to quadrant IV.

Domain:

$\{x|x \in \mathbb{R}\}$

Range:

$\{y|y \in \mathbb{R}\}$

Naomi then used her graphing software to investigate the characteristics of quadratic functions and **cubic functions**. She graphed multiple examples of functions of each degree. Then she sketched some of her examples in a table and recorded her observations.

As Naomi investigated the characteristics of the graphs she created, she considered the following characteristics:

- the number of x -intercepts
- the y -intercept
- the end behaviour
- the domain
- the range

? What are the characteristics of the graphs of quadratic and cubic functions?

Reflecting

- How is the possible number of x -intercepts related to the degree of a polynomial function?
- Naomi claims that all polynomial functions of degree 1, 2, or 3 have only one y -intercept. Do you agree or disagree? Explain.
- Describe how the end behaviour of a polynomial function is related to the degree of the function.
- Describe how the domain and range of a polynomial function are related to its degree.
- Explain why some cubic polynomial functions have **turning points** but not maximum or minimum values.
- Polynomial functions of degree 0 are called constant functions. Describe characteristics of the graphs of constant functions.

Communication *Tip*

Polynomial functions are named according to their degree. Polynomial functions of degrees 0, 1, 2, and 3 are called constant, linear, quadratic, and cubic functions, respectively. The terms in a polynomial function are normally written so that the powers are in descending order. For example,

a constant function, degree **0**: $f(x) = 5x^0$

a linear function, degree **1**: $f(x) = 2x^1 + 1$

a quadratic function, degree **2**: $f(x) = 2x^2 - x + 1$

a cubic function, degree **3**: $f(x) = 2x^3 + 3x^2 - 2x$

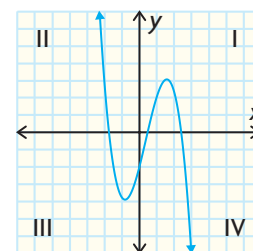
cubic function

A polynomial function of the third degree, whose greatest exponent is three; for example,

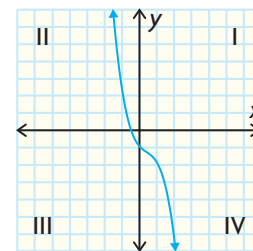
$$f(x) = 5x^3 + x^2 - 4x + 1$$

turning point

Any point where the graph of a function changes from increasing to decreasing or from decreasing to increasing; for example, this curve has two turning points, since the y -values change from decreasing to increasing to decreasing:



This curve does not have any turning points, since the y -values are always decreasing:



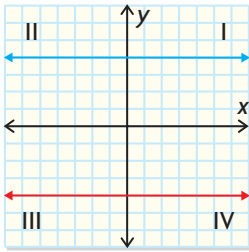
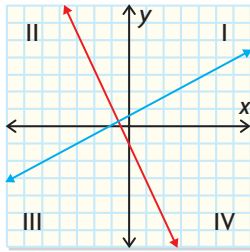
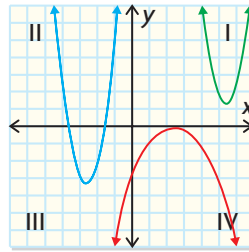
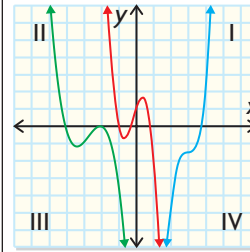
In Summary

Key Ideas

- A polynomial function in one variable is a function that contains only the operations of multiplication and addition, with real-number coefficients, whole-number exponents, and two variables. The degree of the function is the greatest exponent of the function. For example, $f(x) = 6x^3 + 3x^2 - 4x + 9$ is a cubic polynomial function of degree 3.
- The degree of a polynomial function determines the shape of the function.

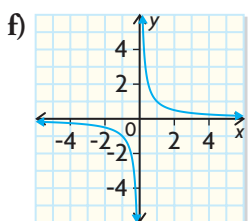
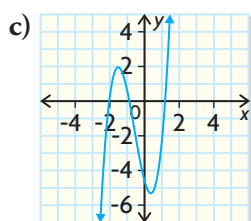
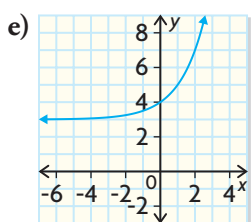
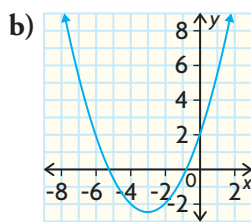
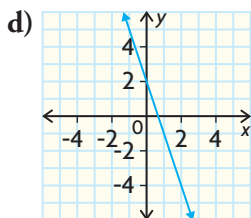
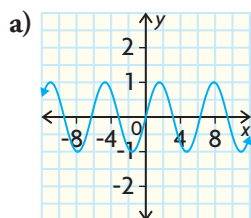
Need to Know

- The graphs of polynomial functions of the same degree have common characteristics.
- The chart below shows sample sketches of functions and displays all the possibilities for the x-intercepts, y-intercepts, end behaviour, range, and number of turning points for each type of function.

Type of Function	constant	linear	quadratic	cubic
Degree, n	0	1	2	3
Sketch				
Number of x-Intercepts	0, except for $y = 0$, for which every point is on the x-axis	1	0, 1, or 2	1, 2, or 3
Number of y-Intercepts	1	1	1	1
End Behaviour	Line extends from quadrant II to quadrant I or from quadrant III to quadrant IV.	Line extends from quadrant III to quadrant I or from quadrant II to quadrant IV.	Curve extends from quadrant II to quadrant I or from quadrant III to quadrant IV.	Curve extends from quadrant III to quadrant I or from quadrant II to quadrant IV.
Domain	$\{x \mid x \in \mathbb{R}\}$	$\{x \mid x \in \mathbb{R}\}$	$\{x \mid x \in \mathbb{R}\}$	$\{x \mid x \in \mathbb{R}\}$
Range	$\{y \mid y = \text{constant}, y \in \mathbb{R}\}$	$\{y \mid y \in \mathbb{R}\}$	$\{y \mid y \leq \text{maximum}, y \in \mathbb{R}\}$ or $\{y \mid y \geq \text{minimum}, y \in \mathbb{R}\}$	$\{y \mid y \in \mathbb{R}\}$
Number of Turning Points	0	0	1	0 or 2

FURTHER Your Understanding

1. Determine which graphs represent polynomial functions. Explain how you decided.



2. Determine the following characteristics for each polynomial function in question 1.

- x -intercepts
- y -intercept
- end behaviour
- domain
- range
- number of turning points

3. Use technology to graph each polynomial function below. Determine the following characteristics for each function:

- number of x -intercepts
- y -intercept
- end behaviour
- domain
- range
- number of turning points

a) $f(x) = 3x - 1$

b) $f(x) = x^2 + 4$

c) $f(x) = -2x^3 - 5x + 3$

d) $f(x) = 5x^3$

e) $f(x) = -3$

f) $f(x) = -(x + 3)^2 + 2$

4. For each type of polynomial function below, sketch graphs to show all possible numbers of x -intercepts.

a) linear

b) quadratic

c) cubic

5.2

Characteristics of the Equations of Polynomial Functions

YOU WILL NEED

- graphing technology

EXPLORE...

- How can you use the equation of this quadratic function to predict the end behaviour of its graph?
 $f(x) = x^2 - 13x + 40$

standard form

The standard form for a linear function is

$$f(x) = ax + b$$

where $a \neq 0$

The standard form for a quadratic function is

$$f(x) = ax^2 + bx + c$$

where $a \neq 0$

The standard form for a cubic function is

$$f(x) = ax^3 + bx^2 + cx + d$$

where $a \neq 0$

GOAL

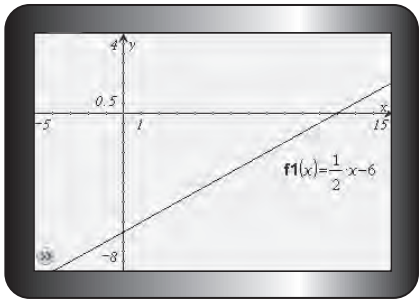
Make connections between the coefficients and constant in the equation of the function and the characteristics of the graph of the function.

INVESTIGATE the Math

The graph of a linear function can be described from its **standard form** equation, using the slope and the y -intercept. The graph of a quadratic function can also be described from its standard form equation, using the y -intercept and the direction of opening.

- ?** How can you predict some of the characteristics of the graph of a cubic polynomial function from its standard form equation?

- A.** The equations of nine polynomial functions are given on the next page. Use technology to draw the graph of each function. Use a table like the one below to record the characteristics of each graph.

Standard Form Equation and Graph of Function	
	a) $f(x) = \frac{1}{2}x - 6$ 
Degree of Polynomial Function	1
Number of x -Intercepts	1
y -Intercept	-6
End Behaviour	Line extends from quadrant III to quadrant I.
Domain	$\{x \mid x \in \mathbb{R}\}$
Range	$\{y \mid y \in \mathbb{R}\}$
Number of Turning Points	0

- a) $f(x) = \frac{1}{2}x - 6$
- b) $f(x) = -5x - 2$
- c) $f(x) = -2x^2 + 2x + 4$
- d) $f(x) = x^2 - 6x + 12$
- e) $f(x) = -2x^3 + 4x^2 - 3x + 1$
- f) $f(x) = 2x^3 + 4x^2 - 3x + 1$
- g) $f(x) = x^3 - 2x^2 - 15x + 36$
- h) $f(x) = x^3 - 8$
- i) $f(x) = -x^3 + 2x^2 + 15x - 10$

- B. How is the **constant term** in a polynomial function related to the y -intercept of the graph of the function?
- C. How does the sign of the **leading coefficient** affect the end behaviour of the graph of each type of polynomial function?
- D. How can you predict some of the characteristics of the graph of a cubic polynomial function from the standard form equation of the function?

leading coefficient

The coefficient of the term with the greatest degree in a polynomial function in standard form; for example, the leading coefficient in the function

$$f(x) = 2x^3 + 7x$$

is 2.

Reflecting

- E. Use a sketch to explain how changing the constant term in a cubic polynomial function can change the number of x -intercepts on the graph of the function.
- F. Why does the sign of the leading coefficient in a polynomial function affect the end behaviour of the graph?
- G. How does the degree of a polynomial function relate to
 - i) the maximum number of x -intercepts the graph may have
 - ii) the maximum number of turning points the graph may have

APPLY the Math

EXAMPLE 1

Reasoning about the characteristics of the graph of a given polynomial function using its equation

Determine the following characteristics of each function using its equation.

- number of possible x -intercepts
 - y -intercept
 - end behaviour
 - domain
 - range
 - number of possible turning points
- a) $f(x) = 3x - 5$
 - b) $f(x) = -2x^2 - 4x + 8$
 - c) $f(x) = 2x^3 + 10x^2 - 2x - 10$



Mary's Solution

a) $f(x) = 3x - 5$

Degree: 1

Number of x -intercepts: 1

y -intercept: -5

End behaviour: The graph extends from quadrant III to quadrant I.

Domain: $\{x \mid x \in \mathbb{R}\}$

Range: $\{y \mid y \in \mathbb{R}\}$

Number of turning points: 0

----- This is a linear polynomial function. Its degree is 1. Polynomial functions of degree 1 have one x -intercept.

----- The constant term of the equation in standard form is -5 , so this is the y -intercept.

----- The function has a positive leading coefficient, so the slope of the line is positive. This means that the line extends from quadrant III to quadrant I.

----- The domain and range of a linear function are all real numbers.

----- A linear function always increases or always decreases, so it has no turning points.

b) $f(x) = -2x^2 - 4x + 8$

Degree: 2

Number of x -intercepts: 2

y -intercept: 8

End behaviour: The graph extends from quadrant III to quadrant IV.

Domain: $\{x \mid x \in \mathbb{R}\}$

Range: $\{y \mid y \leq \text{maximum}, y \in \mathbb{R}\}$

----- This is a quadratic polynomial function. Its degree is 2.

----- Since the function has a negative leading coefficient, I know that the parabola opens down. The constant term is 8, so this is the y -intercept. Since the y -intercept is positive and the parabola opens down, I know that the vertex is above the x -axis. Therefore, the function has two x -intercepts.

----- The parabola opens down, so it extends from quadrant III to quadrant IV.

----- The domain is all real numbers. I can't determine the range without knowing the vertex, and I don't know the vertex without graphing the function. From the equation of the function, I know that the parabola opens down. Therefore, the range is restricted to all y -values less than or equal to the maximum value of the parabola.

c) $f(x) = 2x^3 + 10x^2 - 2x - 10$

Degree: 3

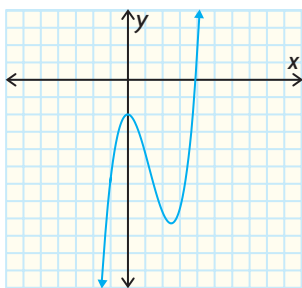
Number of possible x -intercepts: 1, 2, or 3

----- This is a cubic polynomial function. Its degree is 3.

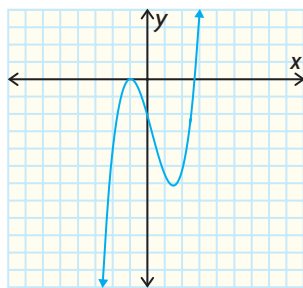
----- Depending on the number of turning points and the location of the function on the grid, a polynomial function of degree 3 can have up to three x -intercepts. I can't determine how many x -intercepts there are from the equation of the function.



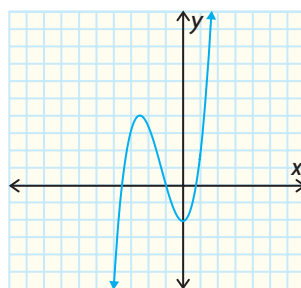
One x -intercept:



Two x -intercepts:



Three x -intercepts:



y -intercept: -10

The constant term of the function in standard form is -10 , so this is the y -intercept.

End behaviour: The graph extends from quadrant III to quadrant I.

The function has a positive leading coefficient. The graph of a cubic function extends from quadrant III to quadrant I when the leading coefficient is positive.

Domain: $\{x \mid x \in \mathbb{R}\}$

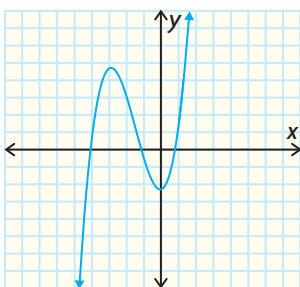
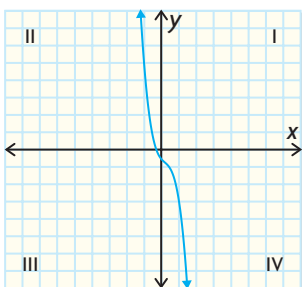
Range: $\{y \mid y \in \mathbb{R}\}$

The domain and range of a cubic function are all real numbers.

Number of possible turning points: 0 or 2

Zero turning points: Two turning points:

A cubic polynomial function can have zero or two turning points. I can't determine how many there are from the equation of the function.



Your Turn

Change the coefficients of the cubic function in part c) to obtain different end behaviour and a positive y -intercept. Verify your answer using technology.

EXAMPLE 2**Connecting polynomial functions to their graphs**

Match each graph with the correct polynomial function.

Justify your reasoning.

$$g(x) = -x^3 + 4x^2 - 2x - 2$$

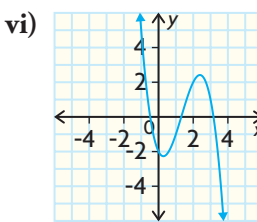
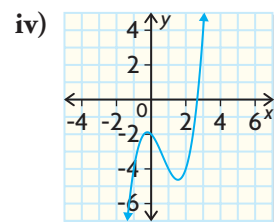
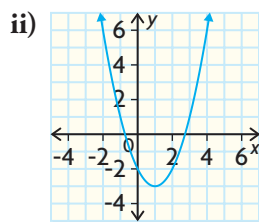
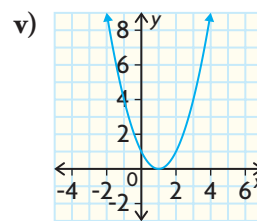
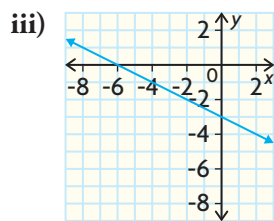
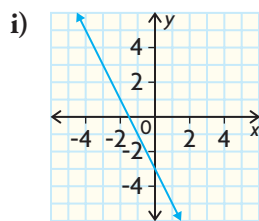
$$j(x) = x^2 - 2x - 2$$

$$p(x) = x^3 - 2x^2 - x - 2$$

$$h(x) = -\frac{1}{2}x - 3$$

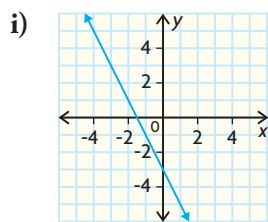
$$k(x) = x^2 - 2x + 1$$

$$q(x) = -2x - 3$$

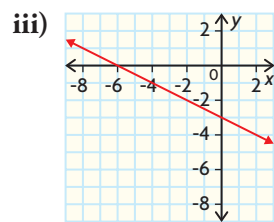
**Jim's Solution**

There are two linear, two quadratic, and two cubic graphs. I grouped together the graphs and equations that represent functions of the same degree.

Linear graphs and equations:



$$q(x) = -2x - 3$$



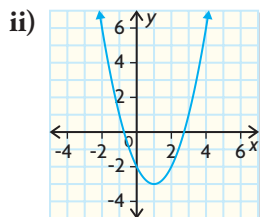
$$h(x) = -\frac{1}{2}x - 3$$

- Both graphs have negative leading coefficients that represent the slope.
- The y -intercept of both graphs is -3 .
- The difference between the two graphs is the steepness of the lines.
Graph i) crosses the x -axis at -1.5 , and graph iii) crosses the x -axis at -6 , so the graph for i) is steeper.

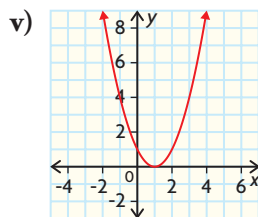
Therefore, graph i) matches $q(x)$ and graph iii) matches $h(x)$.



Parabolic graphs and equations:



$$j(x) = x^2 - 2x - 2$$

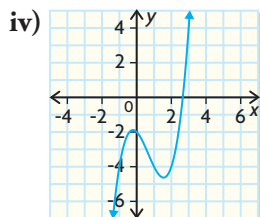


$$k(x) = x^2 - 2x + 1$$

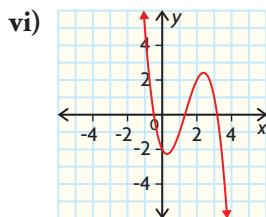
- Both graphs have the same end behaviour, but they have different y -intercepts.
- Graph ii) has a y -intercept of -2 , and graph v) has a y -intercept of 1 .
These y -intercepts are the constant terms in the equations.

Therefore, graph ii) matches $j(x)$ and graph v) matches $k(x)$.

Cubic graphs and equations:



$$p(x) = x^3 - 2x^2 - x - 2$$



$$g(x) = -x^3 + 4x^2 - 2x - 2$$

- These graphs have the same y -intercept of -2 , but they have different end behaviour.
- A cubic function with a positive leading coefficient has a graph that extends from quadrant III to quadrant I. Graph iv) extends from quadrant III to quadrant I, so its function has a positive leading coefficient.
- Using similar reasoning, graph vi) extends from quadrant II to quadrant IV, so the function must have a negative leading coefficient.

Therefore, graph iv) matches $p(x)$ and graph vi) matches $g(x)$.

Your Turn

Describe or draw a sketch to show how each graph in Example 2 would change if the sign of the leading coefficient changed to the opposite sign.

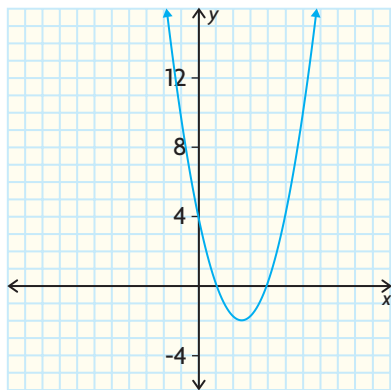
EXAMPLE 3**Reasoning about the characteristics of the graphs of polynomial functions**

Sketch the graph of a possible polynomial function for each set of characteristics below. What can you conclude about the equation of the function with these characteristics?

- a) Range: $\{y \mid y \geq -2, y \in \mathbb{R}\}$
y-intercept: 4
- b) Range: $\{y \mid y \in \mathbb{R}\}$
Turning points: one in quadrant III and another in quadrant I

Leanne's Solution

- a) Range: $\{y \mid y \geq -2, y \in \mathbb{R}\}$
y-intercept: 4
Because the range is a subset of the set of real numbers, I know that this function must be a quadratic function. It has a minimum at $y = -2$.



Because of the end behaviour in my sketch, I know that the equation of the function has a positive leading coefficient.

I also know that the constant term is 4, because this is the y-intercept of the graph.

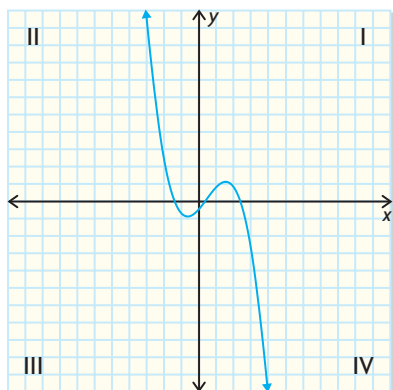
----- Since the quadratic function has a minimum, the parabola opens upward.

----- My sketch satisfies the given characteristics, but I know that there are many other possibilities. For example, the vertex could have been in quadrant III.

b) Range: $\{y \mid y \in \mathbb{R}\}$

Turning points: one in quadrant III and another in quadrant I

The range is all real numbers, so I know that this is a linear or cubic function. There are two turning points, so I know that it is a cubic polynomial function.



I tried sketching different cubic functions with turning points in quadrants III and I, but I noticed that they all had the same shape. I was only able to sketch a cubic function extending from quadrant II to quadrant IV with the given characteristics.

Since the graph of the function extends from quadrant II to IV, the equation must have a negative leading coefficient.

I was not able to determine the y -intercept using the characteristics given, so I do not know what the constant term is. The curve could be on the origin, or it could pass above or below the origin.

Your Turn

Write a possible equation for each function above. Use technology to check and adjust your equation so that it satisfies the given characteristics.

In Summary

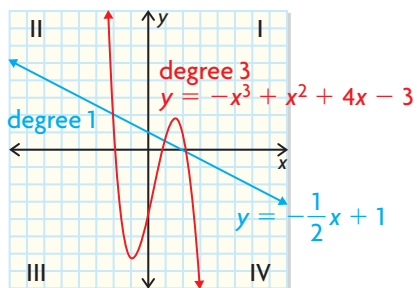
Key Ideas

- When a polynomial function is in standard form:
 - The maximum number of x-intercepts the graph may have is equal to the degree of the function.
 - The maximum number of turning points a graph may have is equal to one less than the degree of the function.
 - The degree and leading coefficient of the equation of a polynomial function indicate the end behaviour of the graph of the function.
 - The constant term in the equation of a polynomial function is the y-intercept of its graph.

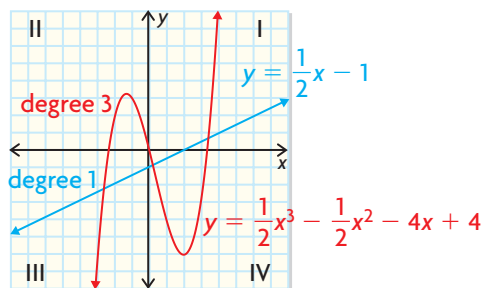
Need to Know

- Linear and cubic polynomial functions with positive leading coefficients have similar end behaviour. Linear and cubic polynomial functions with negative leading coefficients also have similar end behaviour.

If the leading coefficient is negative, then the graph of the function extends from quadrant II to quadrant IV.

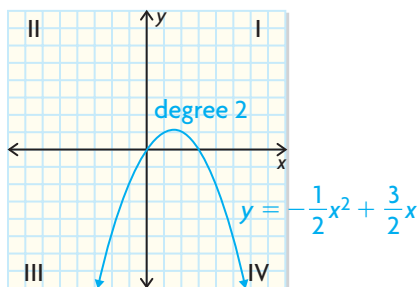


If the leading coefficient is positive, then the graph of the function extends from quadrant III to quadrant I.

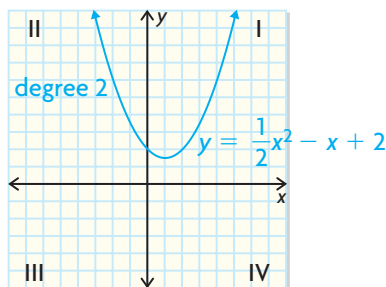


- Quadratic polynomial functions have unique end behaviour.

If the leading coefficient is negative, then the graph of the function extends from quadrant III to quadrant IV.

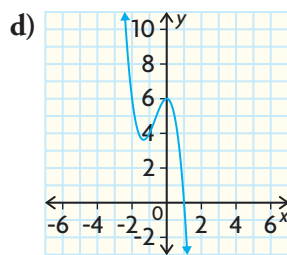
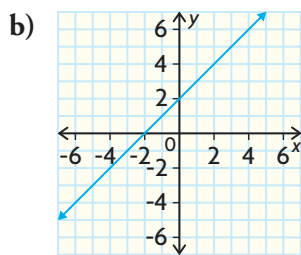
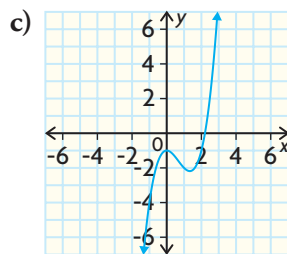
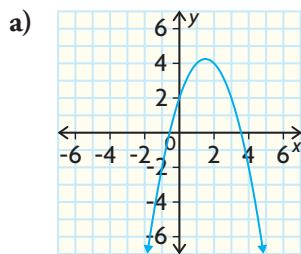


If the leading coefficient is positive, then the graph of the function extends from quadrant II to quadrant I.



CHECK Your Understanding

- Determine the degree, the leading coefficient, and the constant term for each polynomial function.
 - $f(x) = 6x^2 - 3x - 2$
 - $g(x) = -\frac{2}{3}x + 10$
 - $h(x) = -x^3 + 10x + 6$
 - $j(x) = 4x^3 - 2x^2 - 3x - 10$
- For each function in question 1:
 - Determine the minimum and maximum number of x -intercepts of the graph.
 - Determine the end behaviour, domain, and range of the graph.
 - Determine the minimum and maximum number of turning points on the graph.
- Determine the degree, the sign of the leading coefficient, and the constant term for the polynomial function represented by each graph below.



PRACTISING

- For each type of polynomial function below, write an equation that has a y -intercept of 5.
 - constant
 - linear
 - quadratic
 - cubic

5. Describe the end behaviour of each polynomial function.

a) $f(x) = 3x^3 - 2x^2 + x - 1$

b) $g(x) = -x^2 + x + 5$

c) $h(x) = x(x + 2)$

d) $p(x) = 5x + 6 - x^3$

e) $q(x) = -x - 1$

f) $r(x) = \frac{3}{2}x^3 - 2x^2$

6. Match each graph with the correct polynomial function. Justify your reasoning.

i) $y = -x^3 + x + 4$

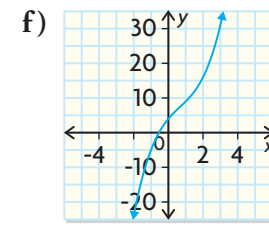
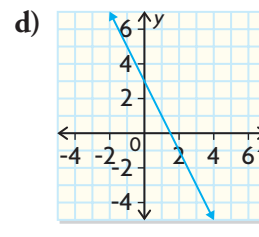
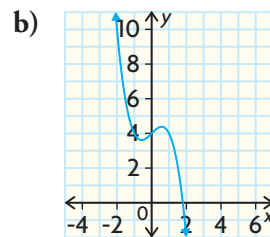
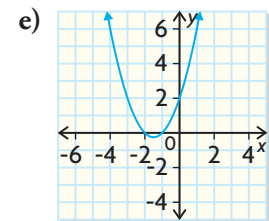
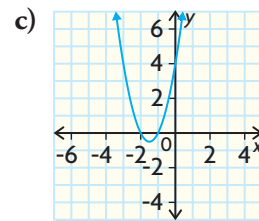
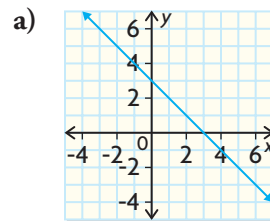
iv) $y = x^3 - 2x^2 + 6x + 4$

ii) $y = 2x^2 + 6x + 4$

v) $y = 3 - x$

iii) $y = (x + 1)(x + 2)$

vi) $y = -2x + 3$



7. Determine the following characteristics of each function.

- number of possible x -intercepts
- y -intercept
- end behaviour
- domain
- range
- number of possible turning points

a) $f(x) = -2x + 5$

b) $v(x) = x^2 + 2x - 6$

c) $u(x) = x^3 - x^2 + 5x - 1$

d) $w(x) = -2x^3 + 4x$

8. Write an equation for a polynomial function that satisfies each set of characteristics.

- a) extending from quadrant III to quadrant IV, one turning point, y -intercept of 2
- b) extending from quadrant III to quadrant I, three x -intercepts
- c) degree 1, increasing function, y -intercept of -3
- d) two turning points, y -intercept of 5
- e) range of $y \leq 2$, y -intercept of 2

9. Lukas described the characteristics of the graph of a polynomial function, but he made an error. Explain and correct Lukas's error.

Lukas's Solution

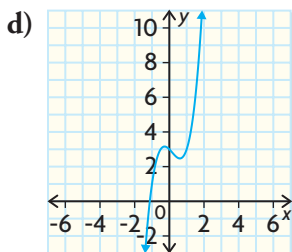
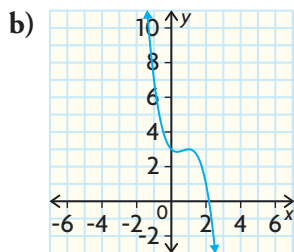
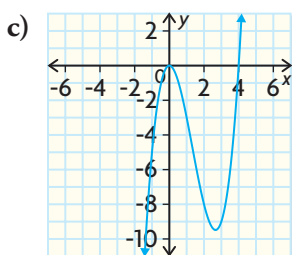
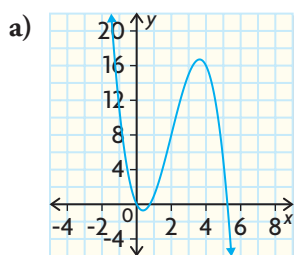
$$f(x) = -3x^3 + 5x^2 + 11x$$

This is a polynomial function of degree 3, so I know that the graph will have zero or two turning points.

The leading coefficient of this cubic function is negative, so I know that the graph extends from quadrant II to quadrant IV.

There is no constant term, so there is no y -intercept. Because this function is cubic, I know that there can be 1, 2, or 3 x -intercepts.

10. Sketch two possible graphs of polynomial functions that satisfy each set of characteristics.
- degree 2, one turning point which is a maximum, constant term of -6
 - two turning points, positive leading coefficient, one x -intercept
 - degree 1, negative leading coefficient, constant term of 10
 - cubic, three x -intercepts, negative leading coefficient
11. Explain why cubic functions may have one, two, or three x -intercepts. Use sketches to support your explanation.
12. Explain why quadratic polynomial functions have maximum and minimum values, but cubic polynomial functions only have turning points.
13. Match each graph with the correct polynomial function. Justify your reasoning.
- $f(x) = -x^3 + 6x^2 - 4x$
 - $g(x) = 2x^3 - x^2 - x + 3$
 - $h(x) = -x^3 + 2x^2 - x + 3$
 - $j(x) = x^3 - 4x^2$



Math in Action

Motion Due to Gravity

For any object that free falls, the graph of height over time is a parabola. The height of an object, h , in metres, after t seconds can be expressed as $h(t) = -4.9t^2 + h_0$ where the constant term, h_0 , represents the initial height of the object.

- In a small group, devise an experiment to determine the height of a window from the ground outside using drops of water.
- Use the leading coefficient, the degree, and the constant term to describe the characteristics of a polynomial function that can model a falling drop of water.
- Find a window that is high above the ground, and carry out your experiment.

14. The average retail price of gas in Canada, from 1979 to 2008, can be modelled by the polynomial function

$$P(y) = 0.008y^3 - 0.307y^2 + 4.830y + 25.720$$

where P is the price of gas in cents per litre and y is the number of years after 1979.

- Describe the characteristics of the graph of the polynomial function.
- Explain what the constant term means in the context of this problem.

15. The tide depth in Deep Cove, British Columbia, from 04:00 to 15:00 on January 6, 2011, can be accurately modelled by the polynomial function

$$f(t) = 0.001t^3 - 0.055t^2 + 0.845t + 0.293$$

where $f(t)$ is the tide depth in metres and t is the number of hours after midnight.

- Use the polynomial function to determine the tide depth at 10:00 on January 6. Compare your result with the actual tide depth in the table to the left.
- Describe the end behaviour of this function.
- Does this function provide accurate tide depths outside the given time frame? Check using the data provided for Jan. 10, 2011.

Time of Day	Tide Depth (m)	
	Jan. 6, 2011	Jan. 10, 2011
00:00	0.8	2.6
01:00	0.8	2.3
02:00	1.2	2.0
03:00	1.8	2.0
04:00	2.6	2.3
05:00	3.5	2.7
06:00	4.2	3.3
07:00	4.7	3.8
08:00	4.8	4.3
09:00	4.6	4.6
10:00	4.2	4.6
11:00	3.8	4.3
12:00	3.5	3.9
13:00	3.2	3.4
14:00	3.2	3.0
15:00	3.4	2.6
16:00	3.6	2.3
17:00	3.8	2.3
18:00	3.9	2.4
19:00	3.8	2.7
20:00	3.4	3.0
21:00	2.9	3.2
22:00	2.2	3.2
23:00	1.6	3.2

Statistics Canada



Closing

16. Suppose that you wanted to describe the graph or equation of a polynomial function and were allowed to ask only three questions about the function. What questions would you ask? Why?

Extending

17. a) Use technology to determine the locations of the turning points of each polynomial function.
- i) $f(x) = x(x - 2)(x - 4)$
 - ii) $g(x) = (x - 6)(x - 2)(x + 1)$
 - iii) $h(x) = -x(x + 3)(x + 6)$
- b) Sketch the graph of each function.
- c) Explain how you could approximate the locations of the turning points without technology.
18. The probability of sinking exactly two out of three free throws with a basketball is given by the function

$$P(x) = 3x^3 - 3x^2$$

where x is the probability of sinking one throw.

- a) Check the reasonableness of the function by evaluating it for different probabilities.
- b) Determine the domain and range of the function.
- c) Determine the coordinates of the turning point of the function within the domain. Explain the meaning of the turning point in the context of this problem.
- d) Determine the x -intercepts, and explain their meaning in the context of this problem.



YOU WILL NEED

- 16 blank playing cards
- 8 playing chips in each of 2 colours
- graphing technology

Applying Problem-Solving Strategies

The Poly–Nomial Game

This is a game for two players. Each player takes eight cards and writes a unique polynomial function on the face of each card. The eight cards should have four cubic functions, two quadratic functions, and two linear functions. On the back of each card, draw a graph of the function. Use technology to help you.

Combine your eight cards with the other player's eight cards. Shuffle all the cards, and lay them out on a four-by-four grid, equation side up.

The Game

- Player A chooses a card and predicts the end behaviour, the y -intercept, and the number of x -intercepts of the function.
- Player B looks at the back of the card to check player A's predictions.
 - If player A's predictions are correct, player B says "Poly!" and turns over the card. Player A places a coloured chip on the card.
 - If player A's predictions are not correct, player B shows the graph and says "NOmial!"
- It is now player B's turn. Players may not select cards that have chips on them or cards that were used in the previous turn of either player.
- The game continues until one player has four chips on a row, a column, or one of the two diagonals, or until it is not possible for either player to win. If neither player can win, the game is tied.

The Strategy

- What strategy can you use to increase your chance of winning if you play first? How does your strategy change if you play second?

Variation

- Switch your set of 16 cards for a set of cards created by another pair of students. Play the game again, with the new set of cards.

FREQUENTLY ASKED Questions

Q: How can you tell whether a given equation represents a polynomial function?

A: A polynomial function is a function in which the coefficients are real numbers and the exponents of the variables are whole numbers. It involves only multiplication and/or addition of real numbers and variables.

For example, consider the following functions:

$$y = 3x^2 - 5x^3 + \frac{1}{2}x \quad y = \sqrt{x} + 4x^2 - 3$$

$$y = \frac{x+3}{2x-5} \quad y = \sqrt{5x^2} + 8x - 10$$

Only two of the above equations are polynomials functions:

$$y = 3x^2 - 5x^3 + \frac{1}{2}x \quad \text{and} \quad y = \sqrt{5x^2} + 8x - 10$$

Q: How can you describe the characteristics of the graph of a polynomial function by looking at the equation of the function?

A: The degree of the function, the sign of the leading coefficient, and the constant term can be used to describe the characteristics of the graph.

- The maximum number of x -intercepts of a polynomial function is equal to its degree.
- The constant term is always the y -intercept of the graph.
- If the function is linear or cubic and the leading coefficient is
 - negative, then the function extends from quadrant II to quadrant IV.
 - positive, then the function extends from quadrant III to quadrant I.
- If the function is quadratic and the leading coefficient is
 - negative, then the graph extends from quadrant III to quadrant IV.
 - positive, then the graph extends from quadrant II to quadrant I.
- All polynomial functions have the domain $\{x \mid x \in \mathbb{R}\}$.
- Linear and cubic functions have the range $\{y \mid y \in \mathbb{R}\}$.
- Quadratic functions have a range restricted by their maximum or minimum value: $\{y \mid y \geq \text{minimum}, y \in \mathbb{R}\}$ or $\{y \mid y \leq \text{maximum}, y \in \mathbb{R}\}$.
- If the function is
 - cubic, then there are no turning points or two turning points.
 - quadratic, then there is one turning point and it is either a maximum or a minimum.
 - linear, then there is no turning point; the function is a line.

Study Aid

- See Lesson 5.1.
- Try Mid-Chapter Review Questions 1 and 2.

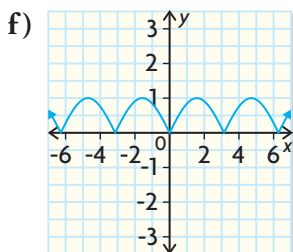
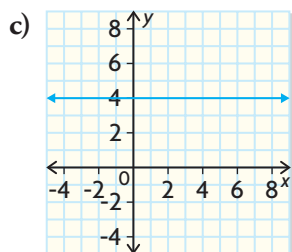
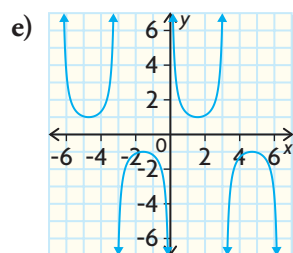
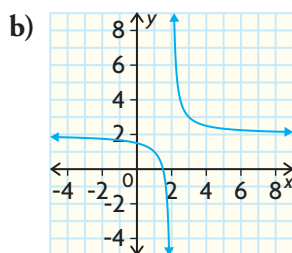
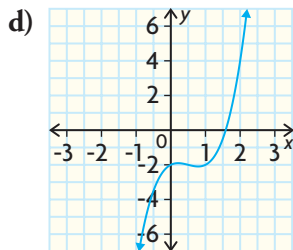
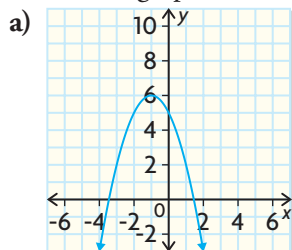
Study Aid

- See Lesson 5.2, Examples 1 to 3.
- Try Mid-Chapter Review Questions 3 to 6.

PRACTISING

Lesson 5.1

1. Which of the following graphs might represent polynomial functions? Explain your reasoning for each graph.

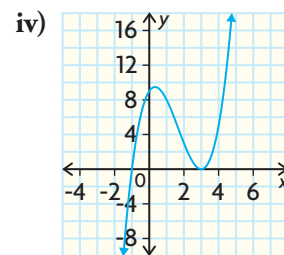
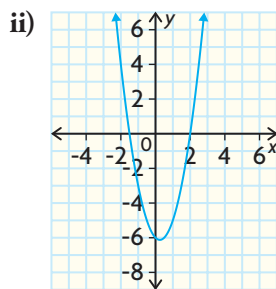
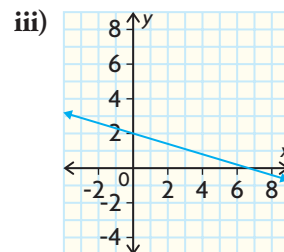
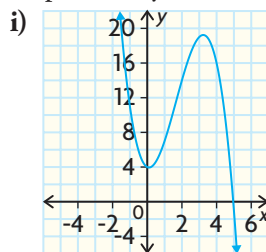


2. a) For each polynomial function in question 1, determine the degree, the domain and range, and the constant term.
 b) The equations of three of the functions in question 1 are listed below. Match each equation with the correct graph, and explain your reasoning.
 i) $f(x) = 2x^3 - 3x^2 + x - 2$
 ii) $f(x) = 4$
 iii) $f(x) = -x^2 - 2x + 5$

Lesson 5.2

3. Describe the end behaviour of each polynomial function using the degree and the leading coefficient.
 a) $f(x) = -x^3 + x^2 + 10$
 b) $f(x) = 3x(x + 2)(x - 1)$
 c) $f(x) = 5x + 6$
 d) $f(x) = -2x^2 + 5$

4. a) Describe the characteristics of each polynomial function shown below. Include the degree, the x -intercepts, the y -intercept, the end behaviour, the domain, range, and number of turning points in your description.



- b) Determine the sign of the leading coefficient and the value of the constant term in the equation of each function.
 5. Using the equation of each function, determine the possible number of x -intercepts, the y -intercept, the end behaviour, the domain, the range, and the possible number of turning points.
 a) $f(x) = -(x - 2)(x + 3)$
 b) $f(x) = x^3 - x^2 + x + 6$
 c) $f(x) = x^2 + 5x - 1$
 d) $f(x) = x^3 - 2x^2 - 8x$
 6. Write a polynomial function that satisfies each set of characteristics. Use technology to check that your function satisfies the characteristics.
 a) extending from quadrant III to quadrant I, two turning points, one x -intercept at the origin
 b) extending from quadrant II to quadrant I, two x -intercepts
 c) extending from quadrant II to quadrant IV, degree 1, y -intercept of -3
 d) one turning point, one x -intercept, y -intercept of 6
 e) range of $y \geq -6$, x -intercepts of 2 and 6

5.3

Modelling Data with a Line of Best Fit

GOAL

Determine the linear function that best fits a set of data, and use the function to solve a problem.

INVESTIGATE the Math

Nathan wonders whether he can predict the size of a person's hand span based on the person's height. His math class investigated this relationship and recorded measurements from 15 students in the tables below.

Height (cm)	Hand Span (cm)	Height (cm)	Hand Span (cm)
165.0	20.0	182.5	25.0
172.5	21.1	172.5	23.0
172.5	17.6	180.0	20.2
153.8	16.5	177.5	21.1
157.5	17.5	165.0	20.7
170.0	19.0	165.0	16.0
168.8	20.8	175.0	21.2
177.5	22.5		

Can you predict a classmate's hand span based on the classmate's height? Model the relationship between the data by writing a linear function that outputs a person's hand span when you input her or his height.

? What linear function best fits the hand-span and height data for high school students?

- Choose the dependent variable, and describe the relationship in the data. What is a reasonable domain and range for this relationship? Explain.
- Plot the data. Use a ruler to draw a line that approximates the trend in your scatter plot.
- Use two points on your line to determine its equation. How does your equation compare with your classmates' equations?

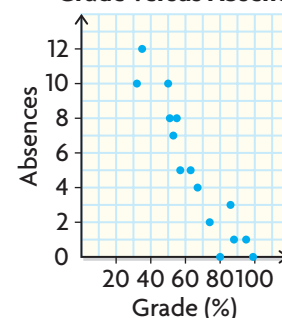
YOU WILL NEED

- graphing technology
- ruler
- metre stick
- graph paper

EXPLORE...

- The **scatter plot** below compares students' absences from math class with the grade they obtained in the course.

Grade versus Absences



Describe the characteristics of a polynomial function that might be used to model the data in the scatter plot.

Communication Tip

The independent variable is the variable that is being manipulated. The dependent variable is the variable that is being observed. The independent variable is always placed on the horizontal axis of a graph.

line of best fit

A straight line that best approximates the trend in a scatter plot.

regression function

A line or curve of best fit, developed through a statistical analysis of data.

- D. Use technology to determine the **line of best fit** for the data. You determined an equation for a line that approximates the trend in the data in part C. How does your equation of the linear **regression function** compare with your equation from part C? How does your new equation compare with your classmates' new equations?
- E. Collect 15 more data points from your classmates. Determine the equation of a new linear regression function for the combined data.
- F. Use all three linear equations you determined to estimate a person's hand span after measuring her or his height. How do your estimates compare?

Reflecting

- G. In part B, you drew a line that approximates the trend in the hand-span data. Explain the reasoning you used to draw the line.
- H. You have three mathematical models for this relationship: a scatter plot, a line of best fit, and the equation of the line of best fit. Which mathematical model do you think provides a more reliable estimate of a person's hand span for their height? Explain.
- I. You determined three different equations for the line of best fit. Which equation do you think is the best to use for estimating someone's hand span? Explain why.

APPLY the Math

EXAMPLE 1

Using technology to determine a linear model for continuous data

The one-hour record is the farthest distance travelled by bicycle in 1 h. The table below shows the world-record distances and the dates they were accomplished.

Year	1996	1998	1999	2002	2003	2004	2007	2008	2009
Distance (km)	78.04	79.14	81.16	82.60	83.72	84.22	86.77	87.12	90.60

International Human Powered Vehicle Association

- a) Use technology to create a scatter plot and to determine the equation of the line of best fit.
- b) **Interpolate** a possible world-record distance for the year 2006, to the nearest hundredth of a kilometre.
- c) Compare your estimate with the actual world-record distance of 85.99 km in 2006.



British Columbian Georgi Georgiev designed and built a human-powered bicycle, the Varna Tempest, which Canadian Sam Whittingham used to break the one-hour record.

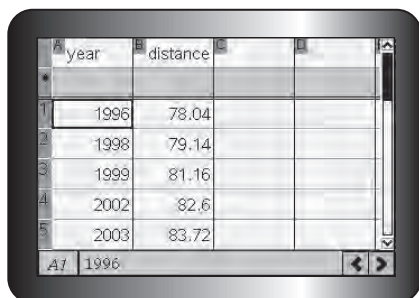
interpolation

The process used to estimate a value within the domain of a set of data, based on a trend.

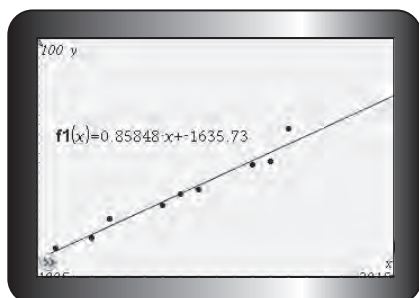
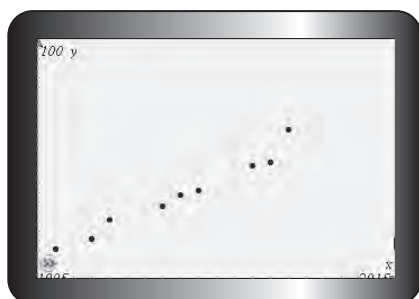


Carmen's Solution: Using a graphing calculator

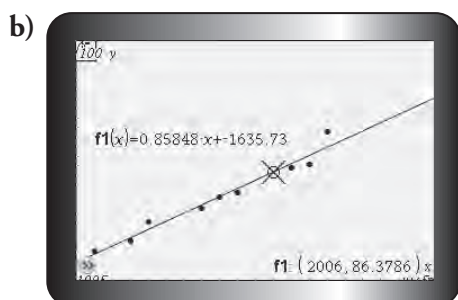
a) I entered the data into my graphing calculator.



year	distance
1996	78.04
1998	79.14
1999	81.16
2002	82.6
2003	83.72



The line of best fit for this data is
 $y = 0.858\,48x - 1635.73$
 where y represents the distance in kilometres
 and x represents the year.



The world record in 2006 may be about
 86.38 km.

I reasoned that year should be the independent variable, since there is a world-record distance for each year. I set my calculator to graph the years on the x-axis and the distances on the y-axis.

I changed my settings to display the x-axis from 1995 to 2015 and the y-axis from 75 to 100.

Based on the scatter plot, the data appears to be linear. I determined the equation of the line of best fit for the data using linear regression. Linear regression gave me the equation of a line that balances points on either side of the line.

I traced to the year 2006 on the graph on my calculator. I obtained 86.3786 km from the graph.

The value of 86.38 km is a reasonable estimate, based on the other world-record distances.

c) $86.38 - 85.99 = 0.38$

My estimate is 0.38 km greater than the actual world-record distance for 2006.

I subtracted the world-record distance from my estimate.

Sandra's Solution: Using a spreadsheet

- a) I used a spreadsheet to determine the equation of the linear regression function.

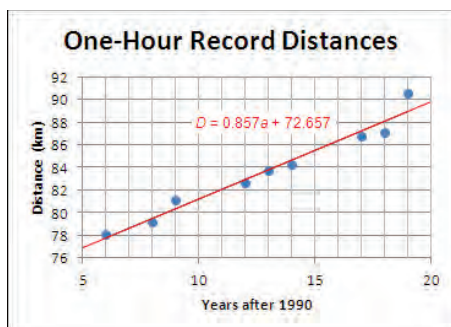
	A	B
1	Years after 1990	Distance (km)
2	6	78.04
3	8	79.14
4	9	81.16
5	12	82.6
6	13	83.72
7	14	84.22
8	17	86.77
9	18	87.12
10	19	90.56

I input the data into the cells. I decided to enter the year data as "Years after 1990" because these values were easier to enter into the spreadsheet.

The distance records change depending on the year in which they were broken, not on the distance achieved, so time is the independent variable.

I created a scatter plot with time along the horizontal axis.

I added the line of best fit to my scatter plot.



I chose a scale that would fit all of my data points.

I noticed that the trend in the scatter plot is somewhat linear.

I chose linear regression in the spreadsheet program to draw the line of best fit and determine the equation of the linear regression function.

The equation that represents the trend is

$$D = 0.857a + 72.657$$

where D represents the distance and a represents the year.



b) $D = 0.857a + 72.657$
 $D = 0.857(16) + 72.657$
 $D = 86.369$

The world record in 2006 may be about 86.37 km.

I used the equation to interpolate a distance for the year 2006. I substituted 16 for a because 2006 is 16 years after 1990.

c) $86.37 - 85.99 = 0.38$
 My estimate is 0.38 km greater than the actual world-record distance in 2006.

I subtracted the world-record distance from my estimate.

Your Turn

If there had been a world-record distance record in the year 2000, what would you expect this distance to have been?

EXAMPLE 2 Using linear regression to solve a problem that involves discrete data

Matt buys T-shirts for a company that prints art on T-shirts and then resells them. When buying the T-shirts, the price Matt must pay is related to the size of the order. Five of Matt's past orders are listed in the table below.

Number of Shirts	Cost per Shirt (\$)
500	3.25
700	1.95
200	5.20
460	3.51
740	1.69



Matt has misplaced the information from his supplier about price discounts on bulk orders. He would like to get the price per shirt below \$1.50 on his next order.

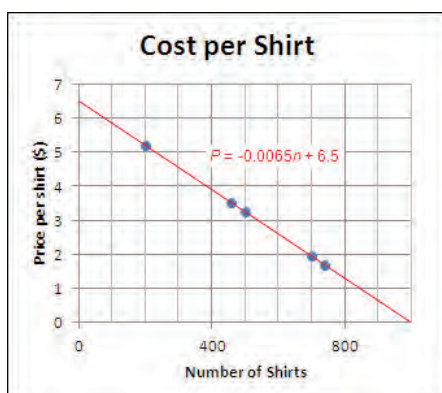
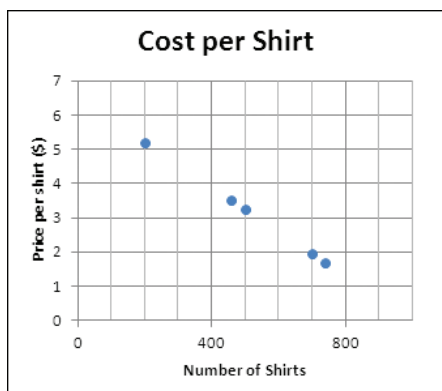
- Use technology to create a scatter plot and determine an equation for the linear regression function that models the data.
- What do the slope and y -intercept of the equation of the linear regression function represent in this context?
- Use the linear regression function to **extrapolate** the size of order necessary to achieve the price of \$1.50 per shirt.

extrapolation

The process used to estimate a value outside the domain of a set of data, based on a trend.

Matt's Solution

a)



Let P represent the price per shirt, and let n represent the number of shirts ordered:

$$P = -0.0065n + 6.5$$

- b) The slope is -0.0065 . It represents a drop in price of \$0.0065 per additional shirt ordered.

The y -intercept is 6.5.

- c) $P = -0.0065n + 6.5$

$$\begin{array}{rcl} 1.50 & = & -0.0065n + 6.5 \\ -5.00 & = & -0.0065n \\ \hline -5.00 & & \\ -0.0065 & = & n \end{array}$$

$$769.230... = n$$

I need to order 770 shirts to get a price less than \$1.50.

I entered the data table into a spreadsheet and used the spreadsheet to create a scatter plot.

Since the price per shirt depends upon the number of shirts ordered, price is the dependent variable. It goes along the vertical axis.

I graphed the line of best fit and obtained the equation of the linear regression function.

The slope represents the rise over the run. On this graph, the rise is the price per shirt and the run is the number of shirts. Therefore, the negative slope means that the price drops for every additional shirt ordered.

The y -intercept is the point on the line where the number of shirts ordered is zero.

I substituted \$1.50 for P in the equation to extrapolate the number of shirts needed for the order.

I cannot order a fraction of a shirt, so I must round up.

Your Turn

Create a problem about Matt ordering T-shirts that you could solve using interpolation. Solve your problem.

In Summary

Key Ideas

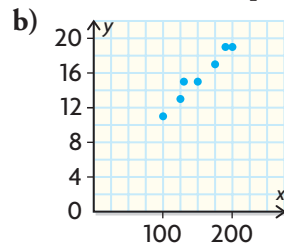
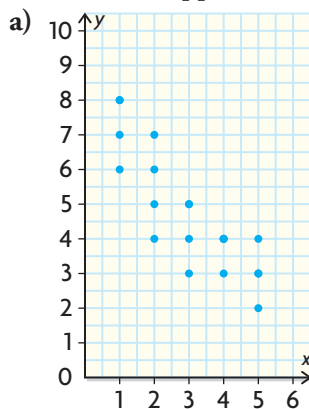
- A scatter plot is useful when looking for trends in a given set of data.
- If the points on a scatter plot seem to follow a linear trend, then there may be a linear relationship between the independent variable and the dependent variable.

Need to Know

- If the points on a scatter plot follow a linear trend, technology can be used to determine and graph the equation of the line of best fit.
- Technology uses linear regression to determine the line of best fit. Linear regression results in an equation that balances the points in the scatter plot on both sides of the line.
- A line of best fit can be used to predict values that are not recorded or plotted. Predictions can be made by reading values from the line of best fit on a scatter plot or by using the equation of the line of best fit.

CHECK Your Understanding

1. Use a clear ruler to help you estimate the slope and y -intercept for a line that best approximates the data in each scatter plot below.

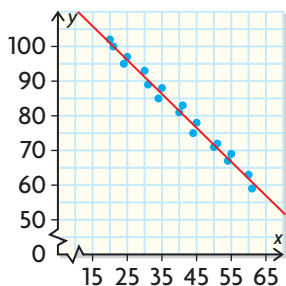


2. Determine the equation of each line in question 1.
3. Determine the independent and dependent variables for each relationship. Justify your reasoning.
 - a) The distance travelled in a car is related to the average speed of the car.
 - b) The size of a family is related to the number of cellphones in the family.
 - c) The number of people in a cafeteria is related to the time of day.
 - d) The number of hours of daylight is related to the time of year.



Yellowknife, Northwest Territories, enjoys the sunniest summers in Canada. There are approximately 1037 h of sunshine during June, July, and August.

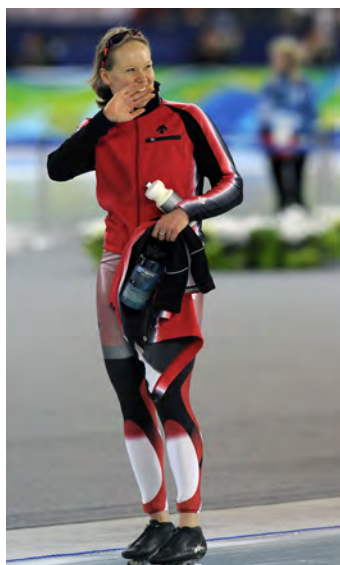
PRACTISING



4. A line of best fit has been drawn for the scatter plot at the left.
 - a) Describe the characteristics of the line of best fit.
 - b) Use the line of best fit to estimate the value of y when x is 47. Is this interpolation or extrapolation? Explain.
 - c) Use the line of best fit to estimate the value of x when y is 70. Is this interpolation or extrapolation? Explain.
 - d) Use the line of best fit to estimate the value of y when x is 15. Is this interpolation or extrapolation? Explain.
5. The world-record times for women's 3000 m speed skating, from 1981 to 2006, are given in the table below.

Skater	Time (min)	Date
Gabi Schönbrunn	4:21.70	March 28, 1981
Andrea Schöne	4:20.91	March 23, 1984
Karin Kania	4:18.02	March 21, 1986
Yvonne van Gennip	4:16.85	March 19, 1987
Gabi Zange	4:16.76	December 5, 1987
Yvonne van Gennip	4:11.94	February 23, 1988
Gunda Kleemann	4:10.80	December 9, 1990
Gunda Niemann	4:09.32	March 25, 1994
Gunda Niemann-Stirnemann	4:07.80	December 7, 1997
Claudia Pechstein	4:07.13	December 13, 1997
Gunda Niemann-Stirnemann	4:05.08	March 14, 1998
Gunda Niemann-Stirnemann	4:01.67	March 27, 1998
Gunda Niemann-Stirnemann	4:00.51	January 30, 2000
Gunda Niemann-Stirnemann	4:00.26	February 17, 2001
Claudia Pechstein	3:59.27	March 2, 2001
Claudia Pechstein	3:57.70	February 10, 2002
Cindy Klassen	3:53.34	March 18, 2006

International Skating Union



Cindy Klassen at the Vancouver Winter Olympics, 2010

- a) Create a scatter plot to compare the world-record time with the year in which it was set.
- b) Describe the characteristics of the trend between the variables.
- c) Determine the equation of the linear regression function for the data. What do the slope and y -intercept of the equation of the linear regression function represent in this context?
- d) Cindy Klassen earned her first world record in 2005, but her time is missing from the table. Interpolate her world-record time in 2005.
- e) Research Cindy's actual world-record time in 2005. How close was your estimate?

6. The world-record time for the men's 100 m sprint was 10.00 s in 1960. The table below shows the world-record times since 1960.

Years after 1960	0	8	23	31	36	39	45	48	49
Time (s)	10.00	9.95	9.93	9.86	9.84	9.79	9.77	9.72	9.58

- Create a scatter plot to display the data.
 - Describe the characteristics of the trend in the data.
 - Determine the equation of the linear regression function for the data. What do the slope and y -intercept of the equation represent in this context?
 - Interpolate a possible world-record time for 2007.
 - Asafa Powell, from Jamaica, accomplished a world-record time on September 9, 2007. Research his time, and determine the difference between this actual time and your estimate.
7. Average daily temperatures for the month of July, from 14 weather stations in British Columbia, are listed in the table below.

Weather Station	Latitude of Weather Station (°)	Mean Temperature for July (°C)
Keremeos	49.2	20.9
Dease Lake	58.4	12.8
Hixon	53.5	16.5
Heffley Creek	50.9	16.8
Lake Cowichan	48.8	17.5
McBride	53.4	15.1
North Vancouver	49.3	16.8
Nakusp	50.3	18.3
Prince George	54.1	15.5
Vanderhoof	54.0	16.3
Likely	52.6	15.4
Bullmoose	55.1	13.4
Fort St. John	56.2	15.7
Todagin Ranch	57.6	11.6

Environment Canada

- What relationship do you expect to see between the latitudes of the weather stations and the mean temperatures? Plot the data to verify your prediction.
- Determine the equation of the linear regression function for the data.
- According to the linear regression function, what average temperature would you expect at a latitude of 52.0°N ?
- At what latitude would you expect a mean temperature of 18°C for July?



Asafa Powell during his record-breaking run at the International Association of Athletics Federation's Grand Prix in Rieti, Italy

8. Call-Me Cellular is rolling out a new marketing plan that is aimed at families. Before finalizing the plan structure, the marketing department needs to know how the number of cellphones in a family is related to the size of the family. The marketing department conducted a survey and collected the following data:

Number of Cellphones	2	2	3	4	1	0	2	2	1	2	1	4	1	3	4	1	0	1
Family Size	5	4	4	5	3	2	4	4	2	2	1	6	4	5	7	2	2	1

- Describe the relationship between the variables in the data.
 - Create a scatter plot, and determine the equation of a linear regression function for the data.
 - The average family in Canada has three people. How many cellphones would you expect to find in an average Canadian family? Explain.
9. Devin is on a budget. He is trying to decide how many graduation events he can afford to attend this year. He interviewed 15 of his older brother's friends to see how much money they spent, compared to the number of events they attended. The data he collected is recorded in the table below.

Number of Events	1	3	4	2	1	3	3	5	4	3	2	4	3	2	2
Money Spent (\$)	200	1300	1500	400	150	1100	900	1500	1450	1100	100	1100	800	300	600

Devin estimates that he has about \$750 to spend on graduation events. Use linear regression to estimate how many events Devin should attend.



10. A large bicycle retailer is planning to open another store. The location that is being considered has 2000 sq ft of floor space. The retailer needs to know the number of bikes that would be needed to stock a store of this size. Use the data from the retailer's other locations to estimate the number of bikes that would be needed.

Number of Bikes	62	58	204	50	190	75	60	60
Floor Space (sq ft)	1500	1250	3200	1200	3000	1600	1550	1300

11. According to Statistics Canada, the life expectancy for Canadians has been increasing over the past few decades.

Years	Life Expectancy (years)	
	Male	Female
1920 to 1922	59	61
1930 to 1932	60	62
1940 to 1942	63	66
1950 to 1952	66	71
1960 to 1962	68	74
1970 to 1972	69	76
1980 to 1982	72	79
1990 to 1992	75	81
2000 to 2002	77	82

- Create two scatter plots: one for the male data and one for the female data.
 - Determine the equation of a linear regression function for each set of data.
 - Use your equations to estimate the life expectancy of males and females in 2012.
12. Mountain Madness Adrenaline Adventures runs mountain-bike tours in Canmore, Alberta. The manager noticed that the number of tours run in a season is related to the value of the Canadian dollar. The following table shows historical data for the average value of the Canadian dollar, as well as the number of tours run in each season.

Year	2002	2003	2004	2005	2006	2007	2008	2009	2010
Dollar Value (\$US)	0.65	0.72	0.76	0.82	0.89	0.95	0.99	0.89	0.96
Number of Tours	18	14	14	13	12	10	8	12	9

Economists predicted that the value of the Canadian dollar would be 1.00 during the summer of 2011. How many tours should Mountain Madness Adrenaline Adventures have expected to run during the summer of 2011? Support your answer with visuals and an analysis of the data.

Closing

13. Suppose that a set of data follows a linear trend.
- What methods could you use to estimate the value of the dependent variable for the data? Give an example.
 - How could you determine the value of the independent variable if you know the value of the dependent variable?



A board foot is a measure of volume. One board foot is equal to 1 ft by 1 ft by 1 in. of lumber.

Extending

14. Martha has been contracted to build several wood cabinets, with proportional door areas, over the summer. She has made a list of the number of board feet of oak that she will need to build each cabinet, based on the width of the cabinet.

Width of Cabinet (in.)	Board Feet Required (bd ft)
12	8.4
15	13.1
20	23.3
25	36.5
30	52.5

- Determine the equation for the linear regression function that models this situation.
- Use your regression equation to determine the number of board feet that Martha will need to build a cabinet with a width of 5 in. and a similar area. Explain why extrapolating does not always make sense.
- Does the data appear linear? Explain.
- What might be a better regression model to use in this situation? Explain why.

5.4

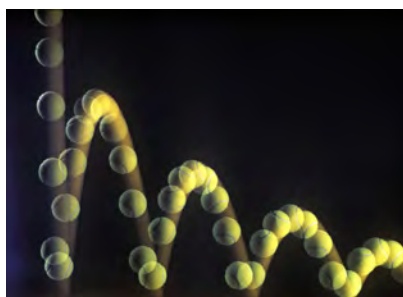
Modelling Data with a Curve of Best Fit

GOAL

Determine the quadratic or cubic function that best fits a set of data, and use the function to solve a problem.

INVESTIGATE the Math

Damien's physics class measured the height of a bouncing ball in the school gymnasium. The ball was repeatedly dropped from the same height. The height was recorded every 0.2 s with a special camera, starting at the first bounce.



Trial 1

Time (s)	0	0.2	0.4	0.6	0.8	1.0
Height (m)	0	1.61	2.43	2.47	1.73	0.20

Trial 2

Time (s)	0	0.2	0.4	0.6	0.8	1.0
Height (m)	0	1.63	2.47	2.53	1.81	0.30

Trial 3

Time (s)	0	0.2	0.4	0.6	0.8	1.0
Height (m)	0	1.60	2.41	2.44	1.69	0.15

? What polynomial function best fits this data?

- Decide which variable is the independent variable. Create a scatter plot by plotting the data from all three trials on the same grid.
- Draw a curve that approximates the trend in the data on your scatter plot.
- Describe the shape of the curve. Include the domain and the range in your description. What type of function does the curve represent? Explain.
- Estimate the coordinates of the vertex.
- Determine the equation of the function, in vertex form, that approximates the data.
- Input the data for all three trials into a calculator or spreadsheet. Determine the equation of the quadratic regression function for the combined data.

YOU WILL NEED

- graphing technology
- ruler
- graph paper

EXPLORE...

- The top of a roller coaster is shown in the photograph below. Describe the characteristics of a polynomial function that might be used to model the shape of this section of the track.



Reflecting

curve of best fit

A curve that best approximates the trend on a scatter plot.

- G. Compare your equation from part E and the equation of the **curve of best fit** from part F. Why are they different?
- H. How could you use one of your functions to determine the time it takes for the ball to reach the second bounce?
- I. How could you use the data or the functions to approximate the maximum height of the ball after the first bounce?
- J. What is the leading coefficient in each function? Why does the coefficient have this value?

APPLY the Math

EXAMPLE 1 Using technology to solve a quadratic problem

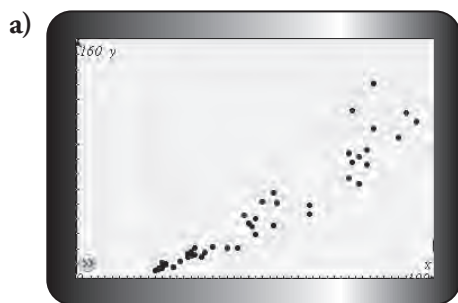
Audrey is interested in how speed plays a role in car accidents. She knows that there is a relationship between the speed of a car and the distance needed to stop. She has found the following experimental data on a reputable website, and she would like to write a summary for the graduation class website.

Speed (km/h)	Distance (m)	Speed (km/h)	Distance (m)	Speed (km/h)	Distance (m)
90	94.4	38	21	83	130.4
36	17	92	111	50	29.1
65	49.2	22	5.6	48	37
56	50.3	31	16.8	45	20.7
65	43.1	50	40	81	86
24	10.9	52	51.2	42	20.6
35	14.2	33	15.9	31	14
55	57.3	27	7.4	38	21
81	76.5	33	20.7	29	11
83	100.3	32	17.9	77	112.3
25	9.1	47	41.9	76	84.1
25	10	95	105.2	55	35.3
77	77.8	24	6.7	79	81.8
32	14.9	23	6.9	23	6.2
76	67.3	79	63.6	49	35



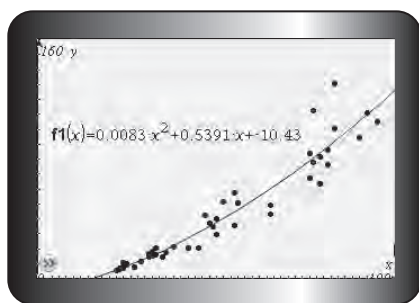
- Plot the data on a scatter plot. Determine the equation of a quadratic regression function for the data.
- Use your equation to compare the stopping distance at 30 km/h with the stopping distance at 50 km/h, to the nearest tenth of a metre.
- Determine the maximum speed that a car should be travelling in order to stop within 4 m, the average length of a car.

Audrey's Solution



I entered the data into my graphing calculator and created a scatter plot. I decided that speed is the independent variable, because it affects the stopping distance, so I placed speed on the horizontal axis.

The relationship shown on the scatter plot is curved, like half a parabola.



I ran the quadratic regression on my calculator, giving me the curve of best fit and its equation.

The equation of the curve of best fit is
 $y = 0.0083x^2 + 0.5391x - 10.43$
 where x represents the speed of a car in kilometres per hour and y represents the stopping distance in metres.

b) $y = 0.0083x^2 + 0.5391x - 10.43$

When $x = 30$ km/h,
 $y = 0.0083(30)^2 + 0.5391(30) - 10.43$
 $y \doteq 13.2$ m

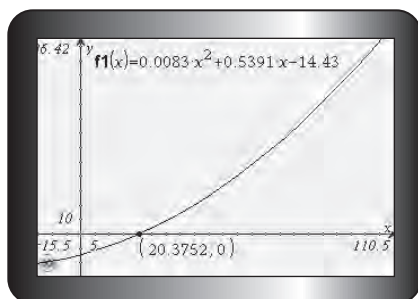
When $x = 50$ km/h,
 $y = 0.0083(50)^2 + 0.5391(50) - 10.43$
 $y \doteq 37.3$ m

The stopping distance is about 3 times greater for a speed of 50 km/h than it is for a speed of 30 km/h.

I substituted the two different speeds into the regression equation to determine the necessary stopping distance.



c) $y = 0.0083x^2 + 0.5391x - 10.43$
 $(4) = 0.0083x^2 + 0.5391x - 10.43$
 $0 = 0.0083x^2 + 0.5391x - 14.43$



The x -intercept is 20.3752.

The car should be travelling at a maximum speed of 20.4 km/h in order to stop within a distance of 4 m, or one car length.

To determine the speed that a car should be travelling in order to stop within 4 m, I substituted 4 for y into the equation and solved for x .

I ended up with a quadratic equation that I could not factor. It is easier to graph the corresponding function than to use the quadratic formula, so I graphed the function.

To determine the roots of the equation, I determined the x -intercepts of the graph of its corresponding function.

There is only one x -intercept in the domain, since $x \geq 0$.

Your Turn

By what factor does the stopping distance increase when the speed is doubled?

EXAMPLE 2 Solving a problem with a cubic regression function

The following table shows the average retail price of gasoline, per litre, for a selection of years in a 30-year period beginning in 1979.

Years after 1979	Price of Gas (¢/L)	Years after 1979	Price of Gas (¢/L)
0	21.98	17	58.52
1	26.18	20	59.43
2	35.63	22	70.56
3	43.26	23	70.00
4	45.92	24	74.48
7	45.78	25	82.32
8	47.95	26	92.82
9	47.53	27	97.86
12	57.05	28	102.27
14	54.18	29	115.29

Statistics Canada

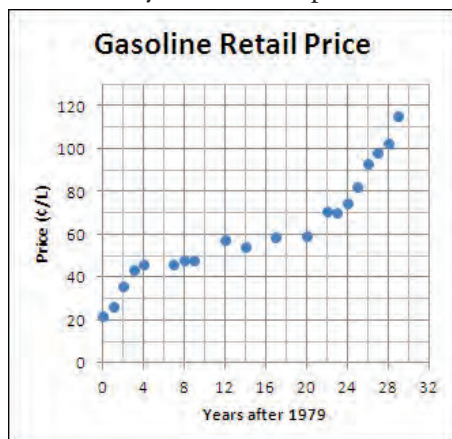
- a) Use technology to graph the data as a scatter plot. What polynomial function could be used to model the data? Explain.



- b) Determine the cubic regression equation for the data. Use your equation to estimate the average price of gas in 1984 and 1985.
- c) Estimate the year in which the average price of gas was 56.0¢/L.

Brad's Solution

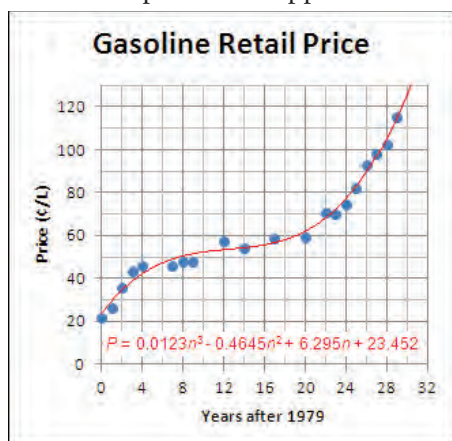
- a) I entered my data into a spreadsheet and created a scatter plot.



----- The price of gas depends on the year, so I placed the price of gas on the vertical axis.

The data is not linear and it appears to extend from quadrant III to quadrant I, so a cubic polynomial function may be a good model.

- b) I used the spreadsheet application to draw a cubic curve of best fit for the data.



The equation of the cubic regression function that models this data is

$$P = 0.0123n^3 - 0.4645n^2 + 6.295n + 23.452$$

where P represents the average price of gas per litre and n represents the number of years after 1979.

For average price of gas in 1984, $n = 5$:

$$P = 0.0123(5)^3 - 0.4645(5)^2 + 6.295(5) + 23.452$$

$$P = 44.9¢/L$$

For average price of gas in 1985, $n = 6$:

$$P = 0.0123(6)^3 - 0.4645(6)^2 + 6.295(6) + 23.452$$

$$P = 47.2¢/L$$

I used the spreadsheet application to determine the equation of the cubic regression function.

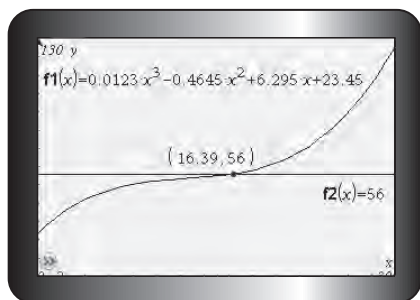
To interpolate the average price of gas in 1984 and 1985, I substituted $n = 5$ and $n = 6$ into the regression equation.

I used 5 and 6 because this is the number of years after 1979.

I estimate that the average price of gas in 1984 was about 44.9¢/L and the average price of gas in 1985 was about 47.2¢/L.

- c) To determine the year in which the average price of gas was 56.0¢/L, I substituted 56.0 for y in the equation of the cubic regression function.

$$56.0 = 0.0123n^3 - 0.4645n^2 + 6.295n + 23.452$$



$$1979 + 16.39 = 1995.39$$

In 1995, the average price of gas was 56.0¢/L.

Since I don't know how to solve a cubic equation algebraically, I used a systems of equations strategy that I have used before with quadratic equations. I graphed each side of the equation as a separate function.

I graphed the functions and determined the n -coordinate of their point of intersection. I know that the n -coordinate of the point of intersection is the solution to the cubic equation.

The n -value for the intersection point is the number of years after 1979 that the price of gas was 56.0¢/L.

I added 1979 to the solution of the equation in order to determine when the price of gas was 56.0 ¢/L.

Your Turn

The actual average prices of gas in 1984, 1989, and 1995 were 69.4¢/L, 72.1¢/L, and 80.1¢/L, respectively. Add these data points to the table, and use interpolation to determine a new average price of gas in 1985.

In Summary

Key Idea

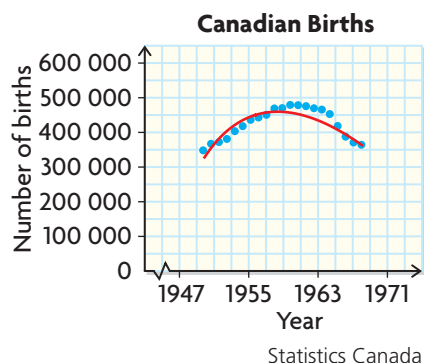
- If the points on a scatter plot seem to follow a predictable curved pattern, then there may be a quadratic or cubic relationship between the independent variable and the dependent variable.

Need to Know

- If the points on a scatter plot follow a quadratic or cubic trend, then graphing technology can be used to determine and graph the equation of the curve of best fit.
- To solve an equation, you can graph the corresponding function of each side of the equation. The x -coordinate of the point of intersection is the solution to the equation.
- Technology uses polynomial regression to determine the curve of best fit. Polynomial regression results in an equation of a curve that balances the points on both sides of the curve.
- A curve of best fit can be used to predict values that are not recorded or plotted. Predictions can be made by reading values from the curve of best fit on a scatter plot or by using the equation of the curve of best fit.

CHECK Your Understanding

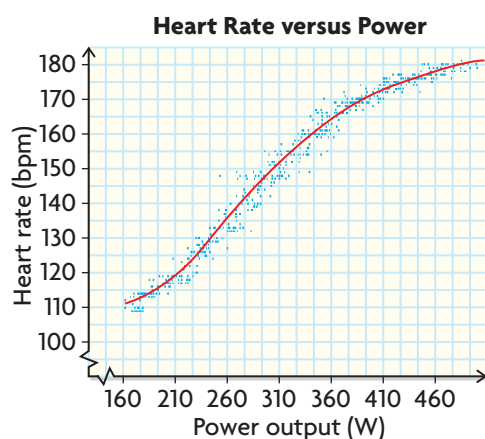
- The following graph shows a quadratic curve of best fit (in red) for the number of Canadian births from 1948 to 1968.



- Describe the trend in the data.
 - Based on the graph, in what year did the greatest number of births occur?
 - How many births took place in 1965?
 - In what years did more than 400 000 births occur?
- The data in the following scatter plot represents several trials of an experiment that was conducted by an athlete to compare his heart rate to his power output. The equation of the cubic regression function for this data is

$$H = -0.000\,002\,4x^3 + 0.001\,930\,2x^2 - 0.224\,951\,6x + 104.820\,761\,9$$

where H represents the athlete's heart rate in beats per minute (bpm) and x represents his power output in watts (W).



- Use the regression equation to estimate the athlete's heart rate when his power output is 310 W.
- Use the curve of best fit to estimate the athlete's power output when his heart rate is 130 bpm.

PRACTISING



3. Josie hit a golf ball from the top of a hill. The height of the ball above the green is given in the table below.

Time (s)	1	2	3	4	5
Height (m)	52.5	73.2	74.6	55.8	16.1

- Describe the characteristics of the data.
 - Determine the equation of the quadratic regression function for the data.
 - Use your equation to determine the height of the ball at
 - 0 s
 - 2.5 s
 - 4.5 s
 - When did the ball hit the ground?
4. A spherical balloon is being inflated. The volume, V , in cubic centimetres is related to the time, t , in seconds.

Volume, V (cm ³)	33.51	113.10	268.08	523.60	904.78
Time, t (s)	0	1	2	3	4

- Use technology to plot the data as a scatter plot. Describe the trend you see.
 - Use cubic regression to create a curve of best fit.
 - Determine the volume of the balloon at 10.5 s.
5. In an experiment, the volume, V , of 1 kg of water is measured as its temperature, T , is increased.

Volume, V (cm ³)	999.87	999.75	1000.01	1000.59	1001.44	1002.52	1003.76
Temperature, T (°C)	0	5	10	15	20	25	30

- Use cubic regression to interpolate the temperature at which the water has the minimum volume.
 - Use cubic regression to extrapolate the volume of the water at a temperature of 40 °C.
6. A dolphin jumped out of the water in a tank and then dove back in. The dolphin's height, in metres, above the water is given in the table below. Using quadratic regression, estimate the maximum height of the dolphin during the jump.

Height (m)	0.59	1.22	1.39	0.73	-1.45
Time (s)	0.5	1.0	1.5	2.5	3.5

7. The fertility rate for Canadian women over 40, for several years after 1977, is shown in the table at the right. The rate is the number of babies born, on average, to a group of 1000 women.

- Use quadratic regression to interpolate two missing data values.
- Compare your interpolated values to the actual values in the table below.

Years after 1977	1	2	7	8	12	13	18	19	20	24	25
Rate per 1000 Females	3.9	3.5	3.1	3.1	3.8	3.9	5.0	5.3	5.4	6.3	6.4

8. A 225 L hot-water tank sprung a leak at $t = 0$ min. The remaining volume was measured every 5 min for the first 40 min.

Volume, V (L)	225	188	155	124	100	72	55	36	23
Time, t (min)	0	5	10	15	20	25	30	35	40

- Plot the data as a scatter plot. Describe the trend.
 - Determine the equation of the quadratic regression function that models the data.
 - Using your equation, determine when the tank was half full.
 - Determine the time when the tank was empty.
9. Statistics Canada recorded the following data for the incidence of lung cancer per 100 000 Canadian males in the years after 1976.
- Use a cubic regression function to model the data and extrapolate the incidence of lung cancer in 2010.
 - According to your polynomial model, when should Canada expect to see the incidence of lung cancer drop below 65 per 100 000 males?

Years after 1976	Cancer Incidence	Years after 1976	Cancer Incidence	Years after 1976	Cancer Incidence
0	75.7	10	96.4	20	82.2
1	78.6	11	95.0	21	79.3
2	85.1	12	95.5	22	80.5
3	83.9	13	93.6	23	79.5
4	83.2	14	92.7	24	77.1
5	91.2	15	90.7	25	75.8
6	92.6	16	90.5	26	73.0
7	95.2	17	91.5	27	70.0
8	97.1	18	86.7	28	69.6
9	93.2	19	84.6		

Years after 1977	Rate per 1000 Females
0	3.9
3	3.3
4	3.4
5	3.3
6	3.1
9	3.3
10	3.5
11	3.7
14	4.0
15	4.3
16	4.6
17	4.8
20	5.4
21	5.4
22	5.8
23	6.1
26	6.8
27	7.2
28	7.4

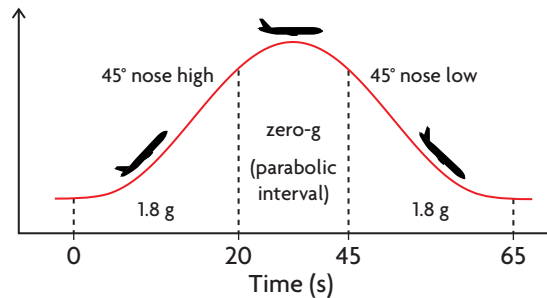
x	y	x	y
0	1	12	69
5	49	15	85
10	75	18	65
15	80	21	82
20	78	24	80
25	83	27	100
30	100	30	105
35	150	33	110
40	200	36	150
50	400	39	190
3	15	42	220
6	55	48	400
9	70	51	450

Closing

10. a) Create a scatter plot for the data in the table at the left.
- b) Determine the equation of the cubic regression function for the data.
- c) Use your equation to determine the y -value when x is 25. How close is your calculated value to the y -value in the table?

Extending

11. To prepare astronauts for the experience of weightlessness, NASA uses versions of an airplane officially known as the Weightless Wonder (and unofficially known as the Vomit Comet). To produce the effect of weightlessness, the aircraft ascends at a 45° angle, transitions into a parabolic flight path, and finishes by descending at a 45° angle. The feeling of weightlessness occurs only during the parabolic interval. For example, during this flight, the feeling of weightlessness lasted for 25 seconds.



The table below shows the time, t , and the altitude, h , from a different Weightless Wonder flight.

t (s)	h (m)	t (s)	h (m)	t (s)	h (m)	t (s)	h (m)
0	7000	8	7926	16	8501	24	7907
1	7116	9	8043	17	8490	25	7791
2	7232	10	8159	18	8459	26	7675
3	7347	11	8265	19	8408	27	7559
4	7463	12	8351	20	8338	28	7443
5	7579	13	8418	21	8248	29	7327
6	7695	14	8465	22	8139	30	7212
7	7811	15	8493	23	8023		

- a) Model the data with a combination of linear and quadratic functions using the appropriate types of regression.
- b) Use your regression models to determine the length of time that the feeling of weightlessness occurred during this flight.

Prime-Generating Quadratic Polynomials

A prime number is a natural number that has only two divisors: 1 and itself. Prime numbers have been studied since ancient times. They are the building blocks of all numbers, since every number is either a prime number or a product of prime numbers. There are eight prime numbers under 20: 2, 3, 5, 7, 11, 13, 17, and 19.

The concept of a prime number is simple to understand, but, even today, mathematicians do not know of a fast way to determine large prime numbers. There is an infinite number of prime numbers, but there is no formula or method for coming up with a prime number.

With the advent of computing and e-commerce, large prime numbers have become increasingly important. A secure way to encrypt information, called RSA public-key encryption, requires very large prime numbers in order to keep data secure. In fact, if anyone ever determined a pattern for the primes that are used for encryption, or even a formula for producing large primes, e-commerce as we know it would crumble.

In 1772, Leonhard Euler discovered a quadratic function that produces a lot of prime numbers:

$$f(x) = x^2 + x + 41$$

This function produces 40 distinct prime numbers for all the integer values of x from 0 to 39. It also excited a lot of mathematicians with the idea that it might be possible to find a formula for all prime numbers.

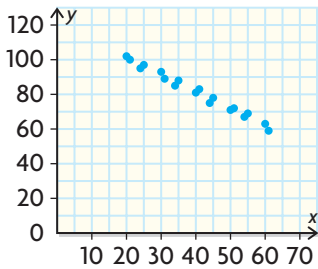
Can you write a polynomial function that produces prime numbers?

- A. Using technology, generate the 40 prime numbers produced by Euler's function and list them in a table.
- B. Determine the value of the function when $x = 40$. How can you tell that this value is not a prime number?
- C. Analyze the function to determine some of its characteristics. Why are these characteristics important for a prime-generating polynomial function?
- D. Design a polynomial function that can produce at least four prime numbers.



The Electronic Frontier Foundation offers prizes for discovering large primes. Nayan Hajratwala was awarded \$50 000 on April 6, 2000, for discovering a 1 000 000 digit prime.

The Clay Mathematics Institute offers \$1 000 000 prizes for solving difficult mathematics problems, which the institute calls the Millennium Prize Problems. One such problem is the proof of the Riemann Hypothesis, which was formulated in 1859 by Bernhard Riemann. This proof gives a pattern for the occurrence of primes within the set of all real numbers.



1. a) Write the equation of a linear function that satisfies each set of characteristics.
 - i) extending from quadrant II to quadrant IV, x -intercept of 3
 - ii) extending from quadrant III to quadrant I, y -intercept of -6
 b) Repeat part a) for a cubic function.
2. Using the equation of each function, determine the possible number of x -intercepts, the y -intercept, the end behaviour, the domain, the range, and the possible number of turning points.
 - a) $f(x) = -(x + 2)^2 + 3$
 - b) $g(x) = \frac{x + 3}{2}$
 - c) $h(x) = (x - 7)(x + 2)(x + 3)$
 - d) $j(x) = -3x^3 + 27x + 2$
3. Write the equation of a polynomial function that satisfies each set of characteristics.
 - a) extending from quadrant II to quadrant IV, two turning points, y -intercept of -6
 - b) extending from quadrant III to quadrant IV, degree 2, x -intercepts of -2 and 4
4. Describe the characteristics of the graph of each polynomial function.
 - a) $g(x) = 2(x - 2)^2 + 4$
 - b) $h(x) = -x^3 + x^2 - x - 1$
5. Describe the characteristics of a line of best fit for the scatter plot at the left.
6. Toby recorded the following odometer measurements during a single trip in his car:

Distance (km)	2	5	8	10	11	15	20	30	33
Time (min)	2	6	10	13	14	19	23	35	40

- a) Create a scatter plot for the data.
- b) Determine the equation of the linear regression function.
- c) Use your equation to estimate
 - i) the time it took for Toby to travel 25 km
 - ii) the possible distance he travelled in 45 min
 - iii) the average speed for the trip
7. A spherical balloon is being inflated. The surface area, A , in square metres, is related to the time, t , in minutes.

Surface Area, A (m ²)	13	28	50	79	113
Time, t (min)	0	1	2	3	4

- a) Create a scatter plot for the data.
- b) Determine the quadratic regression function that models the data.
- c) Use your function to interpolate one value and extrapolate another value.

WHAT DO You Think Now? Revisit **What Do You Think?** on page 273. How have your answers and explanations changed?

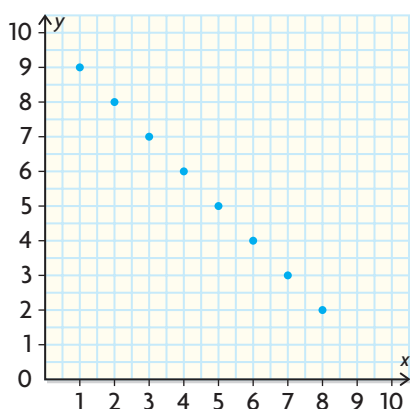
FREQUENTLY ASKED Questions

Q: Why is it better to use the regression function when interpolating from a set of data, rather than just using the data?

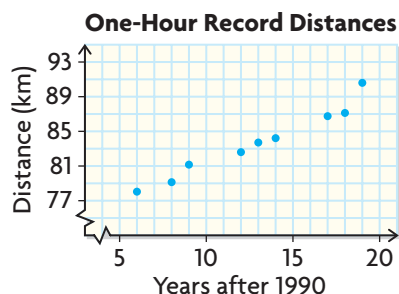
A: Sometimes, just using the data to interpolate may provide a reasonably accurate estimate. It depends on the data. If the data does not have much scatter, then just reading the points may provide an accurate estimate.

Study Aid

- See Lesson 5.3.
- Try Chapter Review Question 6.



However, most real-life data has some scatter. To interpolate from the scatter plot below, you should use regression, which is a statistical analysis. Regression will produce an equation that will minimize the error in your estimate. With technology, you can perform regression easily. You can use the regression equation or line/curve of best fit to interpolate or extrapolate data values.



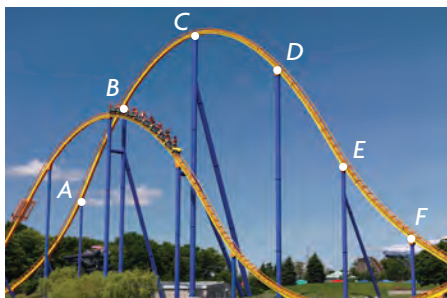
Study Aid

- See Lesson 5.4, Examples 1 and 2.
- Try Chapter Review Questions 7 to 9.

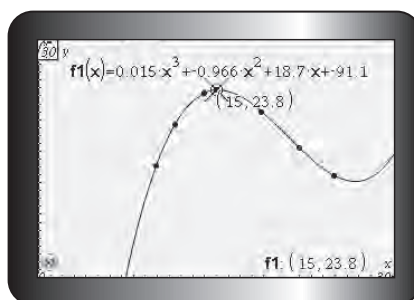
Q: How can you solve a problem involving a data set that can be modelled using a polynomial function?

A: Using a spreadsheet program or the statistical functions on a calculator, you can perform a linear, quadratic, or cubic regression to determine the line or curve of best fit and the equation of the regression function that represents the relationship between the given data. You can interpolate or extrapolate values by tracing along the graph or by substituting values into the equation of the regression function.

For example, the following points are marked on the track below: $A(9.9, 14.1)$, $B(11.5, 19.4)$, $C(14.0, 23.4)$, $D(18.8, 21.0)$, $E(22.0, 16.4)$, and $F(24.9, 12.9)$. In each point, the x -coordinate is the horizontal distance from the point of reference, and the y -coordinate is the vertical distance, both measured in metres. You can use cubic regression to estimate the maximum height of the roller coaster.



First, you need to plot the data and determine the curve of best fit using cubic regression. Then you can trace along the graph to the turning point, where the curve goes from increasing to decreasing.

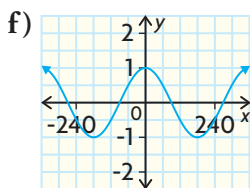
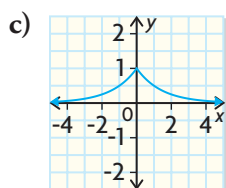
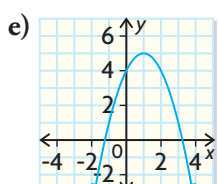
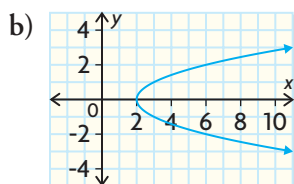
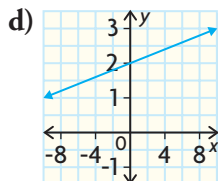
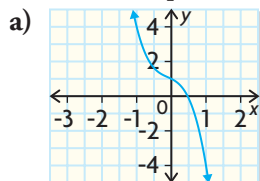


The maximum height of the roller coaster is 23.8 m on this stretch of track.

PRACTISING

Lesson 5.1

- Decide which graphs represent polynomial functions. Explain how you decided.



- For each function that is a polynomial function, use its equation to determine the possible number of x -intercepts, the y -intercept, the domain, the range, and the possible number of turning points.

- $f(x) = \frac{3}{x} + 2$
- $k(x) = x^3 - 4$
- $g(x) = -2(x + 4)^2$
- $h(t) = \sqrt{t^2 - 3}$

Lesson 5.2

- Determine the y -intercept, the end behaviour, the domain, and the range of each polynomial function.

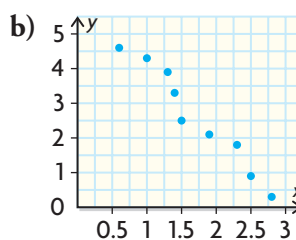
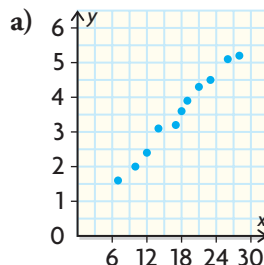
- $f(x) = -x^2 + x - 1$
- $j(x) = 3x^3 - 2x + 5$
- $p(x) = x(x - 6)$
- $r(x) = 2x + 5$

- Describe the characteristics of the graph of each polynomial function.

- $g(x) = -\frac{1}{2}(x + 1)^2 - 4$
- $f(x) = 2x^3 - x^2 - x$

Lesson 5.3

- Estimate the slope and y -intercept of the line of best fit for each scatter plot.



- The cost of a taxi ride in Calgary, for several distances, is given in the table below.

Distance (km)	Cost (\$)
37.95	70.49
2.22	8.40
86.47	152.77
3.55	10.89
19.16	37.76
34.66	64.16
21.17	40.93
12.22	26.55
27.72	52.09

- Create a scatter plot, and describe the relationship that you observe.
- Determine the equation of the linear regression function for the data.
- Use your equation to estimate
 - the cost of a 50 km trip
 - the fixed cost for any taxi trip

Lesson 5.4

7. The height of a roller coaster after cresting its first hill can be represented by the function

$$h(t) = -9.8t^2 + 22$$

where h represents the height in metres and t represents the time in seconds.

- Determine the maximum height of the roller coaster.
- Determine the time it takes the roller coaster to reach half of its maximum height.



8. The following table shows the number of males who entered a trade program in Canada in the odd-numbered years after 1990.

Years after 1990	Number of Males
1	184 705
3	160 020
5	151 945
7	157 875
9	170 710
11	195 220

Statistics Canada

- Create a scatter plot for the data.
- Determine the equation for the quadratic regression function.
- Use your equation to interpolate the values for the even-numbered years.

9. The following table shows the number of females who entered a trade program in Canada in selected years after 1990.

Years after 1990	Number of Females
1	8 245
5	11 425
7	13 305
9	15 675
13	24 280
15	28 755
17	38 070

Statistics Canada

- Determine the equation for the cubic regression function that models the data.
 - Use your equation to estimate the year in which the number of females who entered a trade program was 20 000.
10. A stone is dropped from a bridge into a river below. The table of values shows the time, in seconds, and the height of the stone above the water, in metres.

Time (s)	Position (m)
0.0	20.00
0.5	18.75
1.0	15.00
1.5	8.75
2.0	0.00

- Determine the equation for the quadratic regression function that models the data.
- Use your equation to determine when the stone was 10 m above the water.

Experimenting with Polynomial Models

An Italian physicist named Evangelista Torricelli observed that water flows faster out of a nearly full tank than out of a nearly empty tank. He noticed that the relationship between time and the volume of water in the tank is quadratic. Can a quadratic model that represents the flow of water out of a container be applied to a smaller container?

- ? How does a quadratic model change when an experiment is changed?**

Part 1: Generate the Data

- A. Work in a small group. Find two containers that are different shapes or sizes. Each container must be made from a material that allows you to puncture it with a nail to create a drainage hole. You may use plastic cups, water bottles, juice boxes, or any other containers that can hold from 500 mL to 1000 mL of water. The containers must have an open top.
- B. Start by marking a 50 mL scale along the inside of each container, using a graduated beaker or measuring cups. This scale will help you measure the volume of water as the containers empty.
- C. Puncture the side of each container, near the bottom. Use a nail, 2 or 3 mm in diameter, to make holes that are approximately the same size.
- D. Cover the hole in the larger container, and fill the container with 500 mL of water. Use a basin to collect the water that will drain out.
- E. Before uncovering the hole, decide who will watch the water and indicate when it reaches each 50 mL increment on the scale, who will read the time, and who will record the time.
- F. Uncover the hole, and allow the water to drain into the basin. Collect and record your data.

Part 2: Analyze the Model

- G. Using technology, create a scatter plot and perform quadratic regression for your data. Relate the coefficients and constant in the equation of the regression function to the characteristics of the data in the scatter plot.
- H. Using your model, predict when a smaller container will be half full and when it will be empty.
- I. Repeat steps D to F using a smaller container.
- J. Were your predictions accurate? If not, repeat step G using your data for the smaller container. How do the two models compare?



In 1643, Evangelista Torricelli invented the barometer, a device for measuring atmospheric pressure.

Task Checklist

- ✓ Did you label your graphs?
- ✓ Did you show your work?
- ✓ Did you provide appropriate reasoning?
- ✓ Did you explain your reasoning clearly?

Analyzing Your Data

Statistical tools can help you analyze and interpret the data you collect. You need to think carefully about which statistical tool to use and when, because other people will be scrutinizing your data. A summary of relevant tools is given below.

Measures of Central Tendency

Selecting which measure of central tendency (mean, median, or mode) to use depends on the distribution of your data. As the researcher, you must decide which measure most accurately describes the tendencies of the population. Consider the following criteria when you are deciding which measure of central tendency best describes your set of data.

- Outliers affect the mean the most. If the data includes outliers, use the median to avoid misrepresenting the data. If you want to use the mean, remove the outliers before calculating the mean.
- If the distribution of the data is not symmetrical, but instead strongly skewed, the median may best represent the set of data.
- If the distribution of the data is roughly symmetrical, the mean and median will be close, so either may be appropriate to use.
- If the data is not numeric (for example, colour), or if the frequency of the data is more important than the values, use the mode.

Measures of Dispersion

Both the range and the standard deviation give you information about the distribution of the data in a set.

The range of a set of data changes considerably because of outliers. The disadvantage of using range is that it does not show where most of the data in a set lies—it only shows the spread between the highest and lowest values. The range is an informative tool that can be used to supplement other measures, such as standard deviation, but it is rarely used as the only measure of dispersion.

Standard deviation is the measure of dispersion that is most commonly used in statistical analysis when the mean is used to calculate central tendency. It measures the spread relative to the mean for most of the data in the set.

Outliers can affect standard deviation significantly. Standard deviation is a very useful measure of spread for symmetrical distributions with no outliers.

Standard deviation helps with comparing the spread of two sets of data that have approximately the same mean. The set of data with the smaller standard deviation has a narrower spread of measurements around the mean, and therefore usually has comparatively fewer high or low values.

Normal Distribution and Z-Scores

When working with several sets of data that approximate normal distributions, you can use z -scores to compare the data values. A z -score table enables you to find the area under a normal distribution curve with a mean of zero and a standard deviation of one. To determine the z -score for any data value in a set that is normally distributed, you can use the formula

$$z = \frac{x - \bar{x}}{\sigma}$$

where x is any observed data value, $\bar{x}$ is the mean of the set, and σ is the standard deviation of the set.

Margin of Error and Confidence Level

When analyzing the results of a survey, you may need to interpret and explain the significance of some additional statistics. Most surveys and polls draw their conclusions from a sample of a larger group. The margin of error and the confidence level indicate how well the sample represents the larger group. For example, a survey may have a margin of error of plus or minus 3% at a 95% level of confidence. This means that if the survey were conducted 100 times, the data would be within three percent points above or below the reported results in 95 of the 100 surveys.

The size of the sample that is used for a poll affects the margin of error. If you are collecting data, consider the size of the sample you need for a desired margin of error.

PROJECT EXAMPLE

Choosing tools to analyze your data

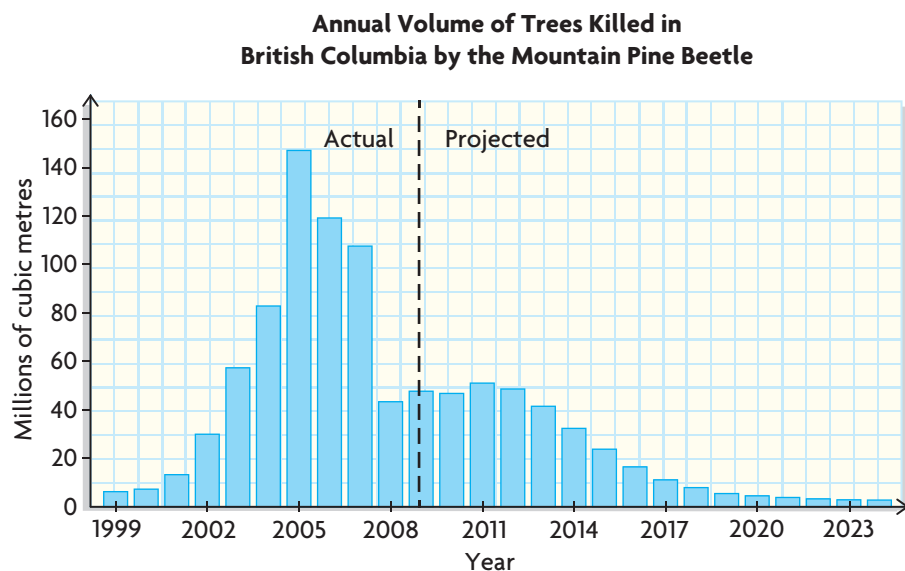
Walther chose as his topic pests and invasive species that cause the greatest damage to natural resources in the Western provinces and territories, and the possible preventive measures that can be taken to control these threats. In his analysis, Walther describes how he determined which statistical tools he might use.



Walther's Analysis

I identified the mountain pine beetle, native to the forests of British Columbia and Alberta but spreading, as one of the most significant pests in these provinces. The European green crab, native to western Europe and northern Africa, is a significant invasive species that threatens the native shellfish that live along the coast of British Columbia. There are many others, including species of plants.

I was able to find data that support my findings. For example, I found the number of hectares that have been damaged in B.C. by the mountain pine beetle for the period 1975 to 2008 at the National Forestry Database, a federal government website. This data is used to make projections about future damage. A graph that I found in an online article, and redrew, shows the actual and projected damage caused by the mountain pine beetle infestation.



I could use a measure of central tendency to represent the average number of hectares damaged or wood damaged over this period. I am not interested in frequency, so the mode is not appropriate. I think the mean would be the best measure to use in this situation. I could also look at the spread in the number of hectares damaged in B.C. over this period using range and standard deviation. The data is not normally distributed, so I do not need to use z-scores. This is secondary data and was not collected by a sampling process, so no margin of error or confidence level needs to be reported.

Your Turn

- Which statistical tools are appropriate for your data? Explain why.
- Use the tools you selected, and calculate the statistics.
- Use these statistics to analyze your data.

4-5

Cumulative Review

1. a) Write an equivalent rational expression for each expression below.

i) $\frac{18a^2}{9a^3}$

ii) $\frac{6x}{x-3}$

- b) State the non-permissible values for each pair of equivalent expressions.

- c) Choose one pair of equivalent expressions from part a).

Demonstrate that they are equivalent by substituting a permissible value into both expressions.

2. a) Determine the non-permissible values for each rational expression.

i) $\frac{2y+1}{2y-1}$

ii) $\frac{3x-6}{6x-12}$

- b) Write two equivalent rational expressions for each expression in part a).

3. Lisa claims that you can use multiplication or division to create equivalent rational expressions. Do you agree? Justify your decision using examples.

4. Determine the non-permissible values for each rational expression. State these values as restrictions, and then simplify the expression.

a) $\frac{30x^5}{-6x^7}$

b) $\frac{5a+10}{10a^2-40}$

5. How is simplifying a rational expression like simplifying a rational number? How is it different? Explain

6. Simplify each of the following expressions.

a) $\left(\frac{6x^2}{5}\right)\left(\frac{15}{8x}\right)$

c) $\frac{12}{(3y)^2} \div 2y$

b) $18a^4 \div \frac{2}{a}$

d) $\left(\frac{36m^3}{6m}\right)\left(\frac{3m^2}{9m^3}\right)$

7. Simplify each expression.

a) $\frac{3a-6}{a+2} \div \frac{a-2}{a+2}$

b) $\left(\frac{9y^2-4}{4y-12}\right)\left(\frac{18-6y}{(3y+2)^2}\right)$

8. Simplify each expression.

a) $\frac{4}{6x^2} + \frac{2}{4x^3}$

b) $\frac{5y}{y+3} - \frac{2y}{y-4}$

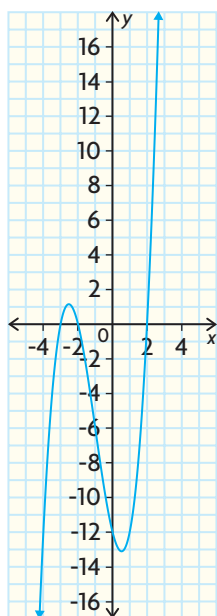
9. State the non-permissible values for each rational equation, and then solve the equation. Verify your solution.

a) $\frac{2}{x} + \frac{5}{3} = \frac{7}{x}$

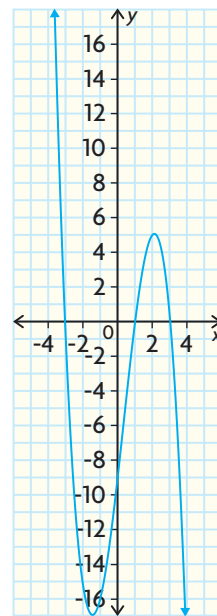
b) $\frac{1}{x+2} + \frac{24}{x+3} = 13$

10. Sid takes 20 min longer to shovel the driveway than his older brother Eric does. If they work together, they can shovel the driveway in 30 min. How long does each person take to shovel the driveway when working alone?
11. Determine the x -intercepts, the y -intercept, the end behaviour, and the domain and range, of each polynomial function shown.

a)



b)



12. For each polynomial function in question 11, determine the degree, the sign of the leading coefficient, the constant term of the equation, and the number of turning points of the function.
13. Determine the following characteristics of each function using its equation.
- | | |
|--------------------------------------|-------------------------------------|
| • number of possible x -intercepts | • domain |
| • y -intercept | • range |
| • end behaviour | • number of possible turning points |
- a) $f(x) = 6x - 2$ c) $u(x) = 2x^3 + 2x^2 - 2x - 2$
- b) $v(x) = -x^2 + 4x - 8$ d) $w(x) = -x^3 + 5x$
14. Sketch two possible graphs of polynomial functions that satisfy each set of characteristics.
- degree 3, positive leading coefficient, two x -intercepts, constant term of 4
 - two turning points, negative leading coefficient, three x -intercepts
 - degree 1, positive leading coefficient, constant term of 3
 - quadratic, negative leading coefficient, two x -intercepts

15. Ananya is a member of the school swim team. The average number of laps that she swims during practice, for various times, is given below.

Time (min)	10	20	30	40	50	60	70	80	90
Laps Completed	22	43	63	81	98	113	126	138	148

- Create a scatter plot to display the data, and determine the linear regression equation of the line of best fit.
 - Use your graph to estimate the number of laps that Ananya can swim in 75 min.
 - Use your regression equation to determine the time that she would need to swim 100 laps.
16. Kim kicks a soccer ball into the air. The following data show the height of the ball at different times during its flight.

Time (s)	0	0.2	0.4	0.7	1.0	1.2	1.5	1.9	2.0
Height (m)	0	1.80	3.20	4.55	5.0	4.80	3.75	0.95	0

- Create a scatter plot to display the data, and determine the quadratic regression equation of the curve of best fit.
 - Use your graph to estimate the times when the ball is at a height of 3.0 m.
 - Use your regression equation to determine the height of the ball at 0.5 s.
17. The amount of energy consumed to heat a building over 15 months is shown in the table at right.
- Create a scatter plot to display the data, and determine the cubic regression equation of the curve of best fit.
 - Use your graph to estimate the time period when the energy consumed to heat the building is less than 150 degree days
 - Use your regression equation to determine the number of degree days needed to heat the building on the middle days of the 7th and 8th months.

Month	Energy Consumed (degree days)
1	226
2	647
3	908
4	965
5	771
6	717
7	283
8	118
9	6
10	0
11	0
12	52
13	221
14	712
15	908